Exercise 15-1 Proof

Suppose that we partition a set of n elements according to the following rule:

For each $k = 0, 1, 2, ..., (p-2)$, thread k will be responsible for rows $\lfloor kn/p \rfloor$ through $\lfloor (k+1)n/p \rfloor - 1$. Thread $(p-1)$ will be responsible for the remaining rows, which will therefore be $\lceil n/p \rceil$.

Lemma 1. *If an array of size n is partitioned into p cells 0,1,2,...,(p-1) such that cell k has elements* $\lfloor (kn)/p \rfloor$ *through* $\lfloor (k+1)n/p \rfloor - 1$*, then each cell has either* $\lfloor n/p \rfloor$ *or* $\lceil n/p \rceil$ *elements.*

Proof. Let $n = qp + r, 0 \leq r < p$. Then the size of cell k, written $s(k)$, is

$$s(k) = \lfloor \frac{(k+1)n}{p} \rfloor - \lfloor \frac{kn}{p} \rfloor = \lfloor \frac{(k+1)(qp+r)}{p} \rfloor - \lfloor \frac{k(qp+r)}{p} \rfloor =$$
$$\lfloor \frac{kqp + kr + qp + r}{p} \rfloor - \lfloor \frac{kqp + kr}{p} \rfloor = (kq + q + \lfloor \frac{kr + r}{p} \rfloor - kq - \lfloor \frac{kr}{p} \rfloor = q + \lfloor \frac{kr + r}{p} \rfloor - \lfloor \frac{kr}{p} \rfloor$$

Since $n = qp + r$, $q = \lfloor (n/p \rfloor$. Thus,

$$s(k) = \lfloor (n/p \rfloor + \lfloor \frac{kr + r}{p} \rfloor - \lfloor \frac{kr}{p} \rfloor$$

Since $0 \leq r < p$ we have

$$0 \leq \lfloor \frac{kr + r}{p} \rfloor - \lfloor \frac{kr}{p} \rfloor \leq \lfloor \frac{kr + p}{p} \rfloor - \lfloor \frac{kr}{p} \rfloor = 1 + \lfloor \frac{kr}{p} \rfloor - \lfloor \frac{kr}{p} \rfloor = 1 \tag{1}$$

Therefore,

$$\lfloor (n/p \rfloor \leq s(k) \leq \lfloor (n/p \rfloor + 1$$

Since $\lfloor (n/p \rfloor + 1 = \lceil n/p \rceil$,

$$\lfloor (n/p \rfloor \leq s(k) \leq \lceil n/p \rceil$$

Note that the $\leq in$ Equation1, is needed. Let $kr = ap + b$ for some a and $0 \leq b < p$. Note that $\lfloor kr/p \rfloor = a$ and $\lfloor b/p \rfloor = 0$. If r is large enough so that $(b + r) \geq p$ then

$$kr + r = ap + b + r \geq ap + p = (a+1)p$$

In other words, $\lfloor (kr + r)/p \rfloor = a + 1$.

On the other hand,

$$\lfloor \frac{kr + p}{p} \rfloor = \lfloor \frac{ap + b + p}{p} \rfloor = \lfloor \frac{ap + p + b}{p} \rfloor = \lfloor \frac{(a+1)p + b}{p} \rfloor = a + 1 + \lfloor b/p \rfloor = a + 1$$

This shows that

$$\lfloor \frac{kr + r}{p} \rfloor = \lfloor \frac{kr + p}{p} \rfloor$$

is possible. □

The difference in the sizes of the cells will always be at most 1 - some will have $\lceil n/p \rceil$ rows, and others $\lfloor n/p \rfloor$ rows. In fact, if $n = qp + r$, $0 < r < p$, then there will be r threads with $\lceil n/p \rceil$ rows and $p - r$ threads with $\lfloor n/p \rfloor$ rows.

Lemma 2. *If an array of size n is partitioned into p cells 0,1,2,...,(p-1) such that cell k has elements $\lfloor (kn)/p \rfloor$ through $\lfloor (k+1)n/p \rfloor - 1$, and cell $p-1$ has elements $\lfloor (kn)/p \rfloor$ through $n-1$, then cell $p-1$ has $\lceil n/p \rceil$ rows.*

Proof. Let $n = qp + r, 0 \leq r < p$. The size of cell $p-1$ is

$$\begin{aligned} s(p-1) &= n - \lfloor \frac{(p-1)n}{p} \rfloor \\ &= qp + r - \lfloor \frac{(p-1)(qp+r)}{p} \rfloor \\ &= qp + r - \lfloor \frac{p(qp+r) - (qp+r)}{p} \rfloor \\ &= qp + r - (qp+r) - \lfloor \frac{-(qp+r)}{p} \rfloor \\ &= -\lfloor \frac{-(qp+r)}{p} \rfloor \end{aligned}$$

The floor of a negative number is the negative of the ceiling of its absolute value, so we have

$$\begin{aligned} s(p-1) &= \lceil \frac{(q+r)}{p} \rceil \\ &= \lceil \frac{np}{p} \rceil \end{aligned}$$

□