7

TELEPORTATION

Hey, hey, hey.
You know, don't be mean.
We don't have to be mean.
'Cuz remember, no matter where you go . . .
there you are.
—Buckaroo Banzai, *The Adventures of Buckaroo Banzai Across the 8th Dimension*, 1984 [169]

In this chapter, we'll meet our first complete quantum algorithm! This algorithm doesn't perform a calculation or give us the answer to a specific problem. Instead, it performs a unique, fascinating task: moving the quantum state of one qubit to another qubit, located anywhere in the universe. And it does this *instantaneously*.

Wait, the no-cloning theorem from Chapter 5 tells us we can't do this, right? The theorem does say that we can't make a copy of a quantum state, but it doesn't prohibit us from *moving* a state from one qubit to another, leaving the original qubit in a different state from how it started.

This process has an exciting name: *quantum teleportation*, or just *teleportation*. It's not quite teleportation the way the term is used in science fiction like *Star Trek*, though. We really do communicate the states of a quantum bit from *here* to *there*, but there are four big differences.

The first is that we're not transferring any kind of matter. We're only communicating the state of a qubit. Even if we transferred the state of enormous numbers of qubits, we still don't have any means for assembling the physical objects that have been put into those states into a grumpy but humane doctor, an exploding warp drive, or even a rock.

The second difference is that we can't send the description of the qubit anywhere we like. We can only transfer the state of a qubit to another qubit it's already been entangled with, and which is already present at the receiving site.

The third difference is that to reliably transfer the state of a quantum bit from one place to another, we must also exchange two classical bits over normal, classical channels, such as radio. That means we can't use this method to share information unless we also share some classical bits over conventional channels.

Finally, the fourth difference is that when we move a quantum state from one qubit to another, the state of the original qubit is changed, and we can't recover its original value.

Given all of these qualifiers, it might be better to call this quantum state *transfer* rather than teleportation. It's not *Star Trek* by a long shot, but transferring the state of one quantum particle to another quantum particle is still pretty cool.

Three features that real quantum teleportation has over the fictional version are that *distance doesn't matter*, *nothing can interfere with the process*, and *the original must be destroyed to be transported*. The first property means that the source and target can be literally anywhere in the universe. The second two properties protect us from ever accidentally creating an "evil Spock" [175].

In this chapter, we'll work a lot with explicit qubit states and the matrices of the operators that modify them. This is unusual. Most of the time, when we analyze a quantum algorithm, we work entirely (or nearly so) with algebra and rarely get down to the level of coefficients. Most of the rest of this book follows that approach. But sometimes working with the actual coefficients can be illuminating, bringing us a little closer to the mechanics of quantum computing. It also allows us to view an algebraic result in a different way, if there aren't too many qubits involved. For these reasons, in this chapter we'll spend most of our time with components, and we'll see explicitly how the operator matrices manipulate the elements of the ket matrices. You won't need to memorize any of these eight-by-eight matrices, as they're all built up from the smaller two-by-two matrices that we're already familiar with.

Okay, enough prep. Let's get teleporting!

The Teleportation Thought Experiment

A great way to think about teleportation is in terms of a story, or what physicists call a *thought experiment*. This story involves two characters. In physics thought experiments with two characters, they are almost always named Alice and Bob, so I'll carry on that tradition here.

In this story, we imagine that Alice and Bob are separated by a great distance: Maybe Alice is on Earth, and Bob is on Mars. Alice has run some algorithm that produces a quantum state, which I'll call the *signal*, described by a ket s with the state $|\sigma\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$, where as always $|\alpha|^2 + |\beta|^2 = 1$. Producing this state is only the first part of a two-part computation. Bob is ready to take over from here and finish the computation, so he needs to have a qubit in the state $|\sigma\rangle$.

Alice could send her physical qubit to Bob, so he can work with it. But let's say that Bob is so remote, and sending things is so slow and expensive, that there's no practical way for Alice to physically send her qubit to Bob.

To get around this limitation, suppose that Bob has taken a qubit named b to Mars. Taken together, Alice's qubit s and Bob's qubit b form a two-qubit system $s \otimes b$. When we think of s and b as a system, it doesn't matter that the qubits are far apart from one another.

Because s is in the state $|\sigma\rangle$, it would be great if there were some sequence of operations that Alice, or Bob, or both of them could follow that would give them the qubit system state $|\sigma\rangle \otimes |\sigma\rangle$. Then Bob's qubit b would also be in the state $|\sigma\rangle$, and they'd have teleported the signal! Unfortunately, this means making a copy of $|\sigma\rangle$, and we know that the no-cloning theorem prohibits that.

Maybe we can avoid cloning if Alice's qubit is changed during teleportation. For example, it might go from $|\sigma\rangle$ to some other state, $|\omega\rangle$. Now if we can put $|\sigma\rangle$ onto Bob's qubit, the new system will be $|\omega\rangle \otimes |\sigma\rangle$. There will be no cloning, and they'll have teleported the signal!

That would be great, but nobody has found a way to do it.

A way that does work requires giving Alice one more qubit. This extra qubit (let's call it a, for auxiliary) will help us perform teleportation.

But how does this help Alice, in her lab on Earth, modify Bob's qubit on Mars? The answer is to link all three qubits together, so that when Alice manipulates her qubits s and a, those operations have an effect on Bob's qubit b.

We know how to do that: Use entanglement! If Alice and Bob created an entangled pair before Bob left, and each kept one qubit of the pair with them, operations on either qubit could affect the other.

We still need some way for Alice to move $|\sigma\rangle$ onto Bob's qubit. The key idea is to create a system state of three qubits that I call the *teleportation state*, which I'll write as $|\tau\rangle$. This is the heart of the whole algorithm. Figure 7-1 shows the teleportation algorithm in two steps.

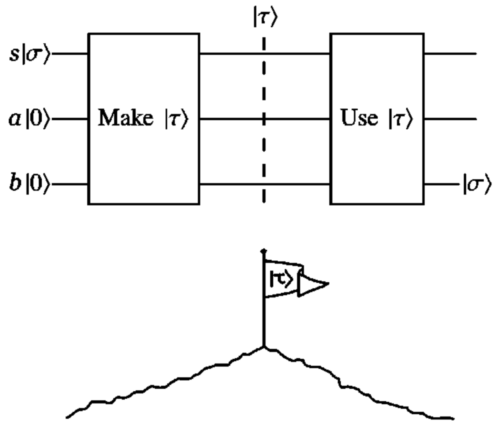


Figure 7-1: The teleportation algorithm can be viewed as two steps.

We can view the first step as climbing a hill to put the qubits into the teleportation state $|\tau\rangle$. We plant our flag (the state $|\tau\rangle$) at the top of the hill and then head down the other side, using $|\tau\rangle$ to put Bob's qubit b into the state $|\sigma\rangle$ that Alice's qubit s was initially in. We don't care about the final states of qubits s and a, so I've left them blank in the figure.

Because the teleportation state $|\tau\rangle$ is at the center of the whole process, let's take a closer look at it.

The Teleportation State $|\tau\rangle$

Let's write the qubits s, a, and b in the teleportation state. Therefore, it has $2^3 = 8$ elements. Creating the teleportation state takes only a few quantum gates, and I'll show you that circuit later in this chapter. For now, I'll ask you to take it on faith that Alice and Bob can create $|\tau\rangle$.

The teleportation state involves the qubits s, a, and b. Let's write these in order from top to bottom, as in Figure 7-2. We'll see that the three qubits are entangled together, so their output is the single entangled state $|\tau\rangle$.

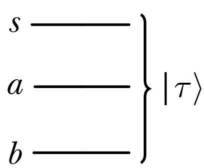


Figure 7-2: The three qubits s, a, and b arranged from top to bottom, making up the entangled state $|\tau\rangle$

The teleportation state $|\tau\rangle$ is an equal superposition of four states. Each is the original $|\sigma\rangle$, perhaps transformed by one or two specific operators.

The state $|\tau\rangle$ is shown in Equation 7.1.

$$|\tau\rangle = \frac{1}{2}\Big(|00\rangle\, I\,|\sigma\rangle + |01\rangle\, X\,|\sigma\rangle + |10\rangle\, Z\,|\sigma\rangle + |11\rangle\, XZ\,|\sigma\rangle \Big) \tag{7.1}$$

At first glance, it looks like we've cloned $|\sigma\rangle$ not just once but three times. But a closer look reveals that there's been no cloning. What we've done is create additional states involving α and β in a single superposition.

We've been doing this kind of thing for several chapters now. For example, suppose we apply an H qugate to $|\sigma\rangle$. The resulting state is shown on the right side of Equation 7.2 (recall our convention that $\vee = 1/\sqrt{2}$).

$$H\,|\sigma\rangle = \vee \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \vee \begin{bmatrix} \alpha + \beta \\ \alpha - \beta \end{bmatrix} \tag{7.2}$$

The final state is an equal superposition of $(\alpha + \beta)\,|0\rangle$ and $(\alpha - \beta)\,|1\rangle$. Both coefficients α and β appear twice, but we haven't cloned anything. Eventually, we'll make a measurement, causing this superposition to collapse, and only one of the states will be associated with the qubit. The same is true of $|\tau\rangle$.

Let's return to $|\tau\rangle$ in Equation 7.1. It's composed of four states in an equal superposition.

In the first state, $|00\rangle\, I\,|\sigma\rangle$, the $|00\rangle$ term refers to the two-qubit state $|0\rangle \otimes |0\rangle$. We then tensor this with $I\,|\sigma\rangle$. This is really just $|\sigma\rangle$, but I included the I for consistency with the other states.

The second state, $|01\rangle\, X\,|\sigma\rangle$, tells us to first form $|0\rangle \otimes |1\rangle$ and then tensor that with $X\,|\sigma\rangle$, or the result of applying the X qugate to the original $|\sigma\rangle$.

The third state, $|10\rangle\, Z\,|\sigma\rangle$, is like the previous one. We first form $|1\rangle \otimes |0\rangle$ and then tensor that with $Z\,|\sigma\rangle$.

Finally, $|11\rangle\, XZ\,|\sigma\rangle$ tensors together $|1\rangle \otimes |1\rangle$ with the state made by applying Z and then X (in that order) to $|\sigma\rangle$ (remember that we read algebraic operators from right to left).

It's the structure of $|\tau\rangle$ that enables teleportation. In Figure 7-2, the first two qubits of $|\tau\rangle$ correspond to s and a and the third to b.

If Alice measures qubits s and a, then as usual she'll get back a single bit for each. Let's say she finds $s = 1$ and $a = 0$. Then the law of partial measurement tells us that the superposition describing the *entire system* must collapse to contain only those states that are consistent with Alice's measurement. There is only one such state in $|\tau\rangle$, $|10\rangle\, Z\,|\sigma\rangle$, and therefore, with certainty, Bob's qubit b is now in the state $Z\,|\sigma\rangle$. Bob knows that Z is its own inverse, or $ZZ = I$, so he can apply Z to this state to get $ZZ\,|\sigma\rangle = |\sigma\rangle$.

Voilá, Bob's qubit b has the state $|\sigma\rangle$. Teleportation achieved!

As promised, there's been no cloning. In order for Alice's signal $|\sigma\rangle$ to make it to Bob's qubit, Alice had to measure both s and a. The process of measuring s collapsed it to either $|0\rangle$ or $|1\rangle$, destroying Alice's copy of $|\sigma\rangle$, thereby enabling us to move $|\sigma\rangle$ to Bob's qubit without cloning.

The Teleportation Process

We can think of teleportation as a four-step process. This will enable us to view the big picture in four smaller chunks that we can then assemble at the end.

The four steps are how Alice and Bob make the teleportation state $|\tau\rangle$, how Alice measures the state of the qubits, how Alice tells Bob which of the four states $|\tau\rangle$ collapsed to, and how Bob applies the correct qugates to get the original $|\sigma\rangle$. Let's take these in order.

Building $|\tau\rangle$

Alice and Bob build the teleportation state $|\tau\rangle$ in three steps. The first step entangles the qubits a and b, the second entangles s with the other two qubits, and the third performs one final step of processing.

Alice and Bob start everything off while they're still together on Earth. After lunch one day, they head to Alice's lab to make two qubits, named a and b, both in state $|0\rangle$. They entangle a and b in the same way that we saw in Figure 5-26. The traditional way to write this in a teleportation circuit is for Bob to apply H to b and then apply a CX using b as a control and a as a target, as shown in Figure 7-3(a).

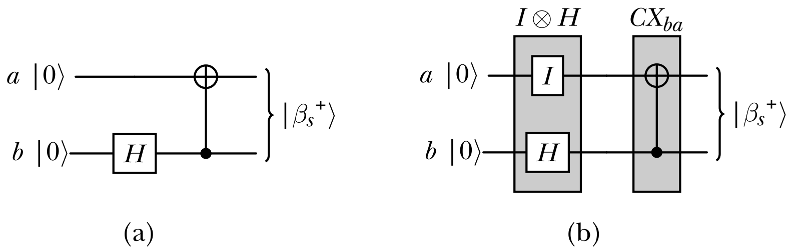


Figure 7-3: (a) Entangling a and b. (b) Explicitly including an identity on a and naming the operator systems.

This CX is drawn upside down compared to how I've usually drawn it before, with the control under the target. We saw this previously in Equation 5.48, where I called it CX'. We confirmed there that this works just as we'd hoped, with the target on the lower line controlling the application of the X qugate on the upper line. In this algorithm, I'll call the qugate CX_{ba} to emphasize that qubit b is controlling qubit a. As a reminder, the matrix form of CX_{ba} from Equation 5.48 is given in Equation 7.3.

$$CX_{ba} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \tag{7.3}$$

Returning to our entanglement step, our experience from Chapter 5 tells us that the output of Figure 7-3 should be $|\beta_s{}^+\rangle$. We can confirm this by first tensoring together $a = |0\rangle$ and $b = |0\rangle$ to make the starting state $|00\rangle$, then modifying them by the system $I \otimes H$ shown in Figure 7-3(b). This qugate system is written out in Equation 7.4.

$$I \otimes H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \vee \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \vee \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \tag{7.4}$$

The second system is CX_{ba}, which we just found. Let's apply both systems to the starting state $|00\rangle$, as shown in Equation 7.5.

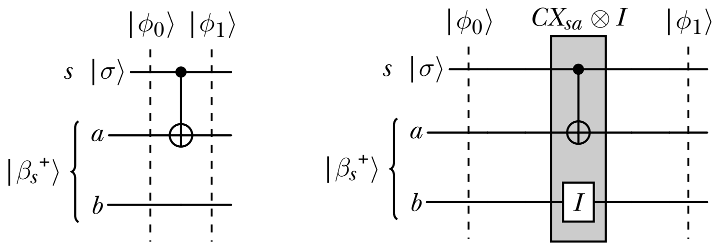
$$\begin{aligned}
CX_{ba}(I \otimes H)|00\rangle &= CX_{ba}((I \otimes H)|00\rangle) && \text{Apply } I \otimes H \text{ first} \\
&= CX_{ba} \left(\vee \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right) && \text{Use } I \otimes H \text{ from Eq. 7.4} \\
&= CX_{ba} \vee \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} && \text{Multiply the matrices} \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \vee \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} && \text{Use } CX_{ba} \text{ from Eq. 7.3} \\
&= \vee \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} && \text{Multiply the matrices} \\
&= \vee(|00\rangle + |11\rangle) = |\beta_s{}^+\rangle && \text{The Bell state } |\beta_s{}^+\rangle
\end{aligned} \tag{7.5}$$

Great! Figure 7-3 does indeed give us the Bell state $|\beta_s{}^+\rangle$.

In our thought experiment, Bob now places qubit b in a special bottle and takes it with him to Mars. Alice also places a in a special bottle and puts it somewhere safe in her lab.

A year passes. One day, Alice completes her experiment, resulting in a qubit named s in the state $|\sigma\rangle$. This is the state she wants to send to Bob.

Alice can only modify s and a, the qubits that she has with her on Earth. In order for Alice to cause operations on the qubits s or a to affect Bob's qubit b far away, qubits s and a need to be entangled with b. Since s is already in a superposition, she can create that entanglement by using s as the control on a CX targeting either a or b. Of these, only a is in the lab with Alice, so she entangles s with a. Because a and b are already entangled, Alice has now created a state where all three qubits are entangled with one another. This second entanglement step is shown in Figure 7-4. I've written CX_{sa} for the usual CX, using s as a control on a.



(a) (b)

Figure 7-4: (a) Entangling s and a. (b) Including the implied I qugate.

Because we're focusing here on matrix elements, let's write out the components of the qubit system $|\phi_0\rangle$, the system just before Alice entangles s with a. This is $|\sigma\rangle$, the state of Alice's qubit s, tensored with $|\beta_s{}^+\rangle$, which we derived in Equation 7.5. The result is shown in Equation 7.6.

$$|\phi_0\rangle = |\sigma\rangle \otimes |\beta_s{}^+\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \otimes \vee \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \vee \begin{bmatrix} \alpha \\ 0 \\ 0 \\ \alpha \\ \beta \\ 0 \\ 0 \\ \beta \end{bmatrix} \tag{7.6}$$

Now we'll entangle s with a. Figure 7-4(a) shows using s as a control on a. As usual, it omits the identity qugate I we could place on the b line. But although it's not in the picture, that identity must be in our operator!

This is because a and b are entangled. Therefore, we have to treat the qubit system as a single system. Any modifications to this qubit system must be described by a single-operator system that accounts for all three qubits. This means we're applying the system $CX_{sa} \otimes I$.

Let's write out this matrix, as shown in Equation 7.7. Because CX_{sa} applies a control on the topmost line to a target immediately below it, its matrix is the familiar CX matrix.

$$CX_{sa} \otimes I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \tag{7.7}$$

Now we can apply the system $CX_{sa} \otimes I$ to the state $|\phi_0\rangle$ we found in Equation 7.6 to get $|\phi_1\rangle$, as shown in Equation 7.8.

$$|\phi_1\rangle = (CX_{sa} \otimes I)\,|\phi_0\rangle = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & k & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \vee \begin{bmatrix} \alpha \\ 0 \\ 0 \\ \alpha \\ \beta \\ 0 \\ 0 \\ \beta \end{bmatrix} = \vee \begin{bmatrix} \alpha \\ 0 \\ 0 \\ \alpha \\ 0 \\ \beta \\ \beta \\ 0 \end{bmatrix} \tag{7.8}$$

I'll be pragmatic now. Looking at $|\phi_1\rangle$ in Equation 7.8, what would it take to turn this into the teleportation state $|\tau\rangle$ in Equation 7.1?

The trick is to write out $|\tau\rangle$ as a single state and compare it to $|\phi_1\rangle$ in Equation 7.8. Then we'll see if we can find a sequence of operations that juggle around the elements of $|\phi_1\rangle$ so that they match $|\tau\rangle$.

I'll find this explicit form of $|\tau\rangle$ in two steps. First, I'll expand each basis state $|00\rangle$ through $|11\rangle$ into its corresponding four-element ket, and then I'll replace each modified version of $|\sigma\rangle$ with the coefficients of its matrix.

As Equation 7.1 shows, we'll need four transformations in all, one for each basis state.

Equation 7.9 summarizes those four transformations for reference. In the fourth row, I applied Z and X operators (in that order) to make the combined operator XZ.

$$\begin{aligned} I\,|\sigma\rangle &= \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}\alpha\\ \beta\end{bmatrix} &= \begin{bmatrix}\alpha\\ \beta\end{bmatrix} \\ X\,|\sigma\rangle &= \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}\alpha\\ \beta\end{bmatrix} &= \begin{bmatrix}\beta\\ \alpha\end{bmatrix} \\ Z\,|\sigma\rangle &= \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}\alpha\\ \beta\end{bmatrix} &= \begin{bmatrix}\alpha\\ -\beta\end{bmatrix} \\ XZ\,|\sigma\rangle &= \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\begin{bmatrix}\alpha\\ \beta\end{bmatrix} &= \begin{bmatrix}-\beta\\ \alpha\end{bmatrix} \end{aligned} \tag{7.9}$$

With these in hand, let's rewrite $|\tau\rangle$. The steps are in Equation 7.10. The first line repeats $|\tau\rangle$ from Equation 7.1. The second line expands the basis states into kets, and the third replaces each modified version of $|\sigma\rangle$ with its state from Equation 7.9. In this last line, the rules of operator precedence tell us to perform the tensor operations before the additions.

$$\begin{aligned} |\tau\rangle &= \frac{1}{2}\big(\,|00\rangle\, I\,|\sigma\rangle + |01\rangle\, X\,|\sigma\rangle + |10\rangle\, Z\,|\sigma\rangle + |11\rangle\, XZ\,|\sigma\rangle\,\big) \\ &= \frac{1}{2}\left(\begin{bmatrix}1\\0\\0\\0\end{bmatrix} I\,|\sigma\rangle + \begin{bmatrix}0\\1\\0\\0\end{bmatrix} X\,|\sigma\rangle + \begin{bmatrix}0\\0\\1\\0\end{bmatrix} Z\,|\sigma\rangle + \begin{bmatrix}0\\0\\0\\1\end{bmatrix} XZ\,|\sigma\rangle\right) \\ &= \frac{1}{2}\left(\begin{bmatrix}1\\0\\0\\0\end{bmatrix} \otimes \begin{bmatrix}\alpha\\ \beta\end{bmatrix} + \begin{bmatrix}0\\1\\0\\0\end{bmatrix} \otimes \begin{bmatrix}\beta\\ \alpha\end{bmatrix} + \begin{bmatrix}0\\0\\1\\0\end{bmatrix} \otimes \begin{bmatrix}\alpha\\ -\beta\end{bmatrix} + \begin{bmatrix}0\\0\\0\\1\end{bmatrix} \otimes \begin{bmatrix}-\beta\\ \alpha\end{bmatrix}\right) \end{aligned} \tag{7.10}$$

Finally, let's explicitly compute the tensor products in the last line of Equation 7.10, giving us Equation 7.11.

$$|\tau\rangle = \frac{1}{2}\left(\begin{bmatrix}\alpha\\ \beta\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\end{bmatrix} + \begin{bmatrix}0\\ 0\\ \beta\\ \alpha\\ 0\\ 0\\ 0\\ 0\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 0\\ 0\\ \alpha\\ -\beta\\ 0\\ 0\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 0\\ 0\\ 0\\ 0\\ -\beta\\ \alpha\end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix}\alpha\\ \beta\\ \beta\\ \alpha\\ \alpha\\ -\beta\\ -\beta\\ \alpha\end{bmatrix} \tag{7.11}$$

Great! Now we have an eight-element ket for $|\tau\rangle$. This is the very same τ we originally saw in Equation 7.1, but it's now represented as a single state vector.

This $|\tau\rangle$ is our goal. We want to turn the $|\phi_1\rangle$ in Equation 7.8 into this $|\tau\rangle$. How can we do this? Is there some operator A (or some sequence of operators that we can multiply together to make A) that we can plug into Equation 7.12 to do the trick?

$$A\,|\phi_1\rangle = |\tau\rangle, \quad \text{or} \quad \vee\, A \begin{bmatrix} \alpha \\ 0 \\ 0 \\ \alpha \\ 0 \\ \beta \\ \beta \\ 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \alpha \\ \beta \\ \beta \\ \alpha \\ \alpha \\ -\beta \\ -\beta \\ \alpha \end{bmatrix} \tag{7.12}$$

Happily, we can indeed build an operator A that does just what we want. If you write down Equation 7.12 with a big empty matrix for A, then you can work through each element and fill in the entries. You'll find some elements must be 1, others −1, and still others must be 0. And to turn the $\vee$ in $|\phi_1\rangle$ into the 1/2 in $|\tau\rangle$, the matrix will need to include another factor of $\vee$. The resulting matrix is shown in Equation 7.13.

$$\vee \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \end{bmatrix} = \left[\begin{array}{c|c} I_4 & I_4 \\ \hline I_4 & -I_4 \end{array}\right] \tag{7.13}$$

You can pull this pattern apart, as shown by the block matrix on the right. In the upper-left, upper-right, and lower-left blocks, we have the four-by-four identity matrix given by $I_4 = I \otimes I$. In the bottom-right block, we have its negative, $-(I \otimes I)$. This pattern of positives and negatives is just what we get from forming $H \otimes (I \otimes I)$. The parentheses aren't needed, but I've put them there to emphasize that we're thinking of $I \otimes I$ as one matrix that gets tensored with H.

In other words, the matrix in Equation 7.13 that takes us from $|\phi_1\rangle$ to $|\tau\rangle$ is given by $H \otimes I_4$, or $H \otimes I \otimes I$. In the circuit diagram, we draw an H on the top line and usually just imply the I qugates, as in Figure 7-5(a). Figure 7-5(b) shows what it looks like with the I qugates drawn explicitly.

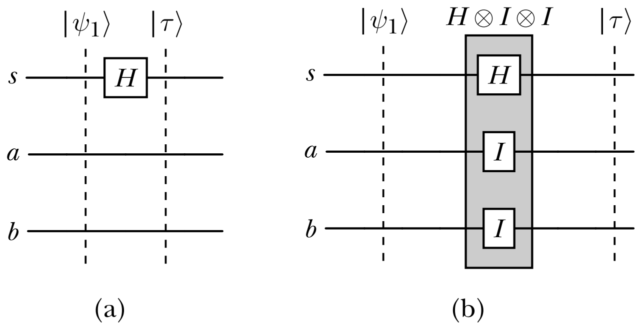


Figure 7-5: (a) Applying the final H to the top qubit, giving us the teleportation state $|\tau\rangle$. (b) Including the implied I qugates.

Because these qubits are entangled, it's not enough to just apply H to s. We must apply the whole system, $H \otimes I \otimes I$, to the entire three-qubit system represented by $|\phi_1\rangle$.

Let's step back for a moment and ask if this H makes sense. It does if you see this H as doing the same job as the first H in the protocol: It's creating a superposition. In this case, it's taking the two-state superposition $|\phi_1\rangle$ from Equation 7.8 into the four-state superposition $|\tau\rangle$ that we want.

Let's put it all together. Starting with qubits a and b in the state $|0\rangle$, we entangle them with an H and CX_{ba}, then we entangle s with those using CX_{sa}, and finally we apply an H to s to create the teleportation state $|\tau\rangle$. The process is shown in Figure 7-6.

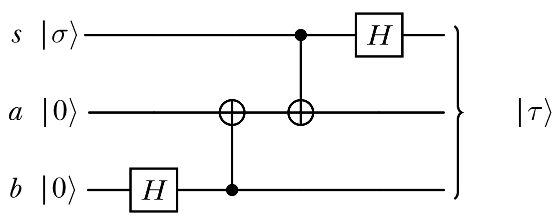


Figure 7-6: The full setup step for teleportation

This is the conceptual peak of the argument: We've created the teleportation state $|\tau\rangle$. The hard work is done! Now we're on the downhill slope from Figure 7-1.

Alice Measures Her Qubits

Now that the teleportation state has been set up, Alice will collapse it to just one state. This will push the state $|\sigma\rangle$ onto Bob's qubit (because it will have nowhere else to go) and simultaneously collapse the qubit s that has held $|\sigma\rangle$ until now.

Alice can measure her two qubits s and a in either order, or even at the same time. Appending this measurement to our existing circuit from Figure 7-6 gives us Figure 7-7. I'm labeling the output bits with the letter m

(for measurement) rather than my usual b (for bit) because we're already using b for Bob's qubit.

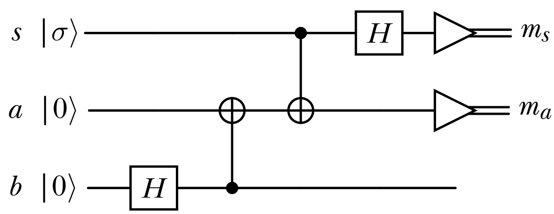


Figure 7-7: After the circuit of Figure 7-6, Alice measures her qubits.

Let's write the measured values from this system as a bitstring $m_s m_a$.

Suppose that Alice measures $m_s = 0$ and $m_a = 1$, or the bitstring 01. The law of partial measurement says that the system state $|\tau\rangle$ must collapse to states that are consistent with this measurement. That is, the system collapses to include only those states that start with $|01\rangle$. There is only one such state in $|\tau\rangle$ from Equation 7.1, and that's $|01\rangle X|\sigma\rangle$. So in this case, after the measurement, Alice's qubits a and s are now $|0\rangle$ and $|1\rangle$ respectively, and Bob's qubit b must be $X|\sigma\rangle$.

Alice Tells Bob the Measurements

When Alice's measurements are complete, our three-qubit system has collapsed to one of the four states in Equation 7.1. That is, depending on what Alice measured, Bob is holding a qubit that is in the state $I|\sigma\rangle$, *or* in the state $X|\sigma\rangle$, *or* in the state $Z|\sigma\rangle$, *or* in the state $XZ|\sigma\rangle$.

If Bob can determine the state of his qubit, he can apply the correct qugates to leave him with $|\sigma\rangle$. So the big question is, how can Bob tell which state he has?

That depends on what Alice measured. So the new question is, how can Bob determine what Alice saw on her meters?

The answer is that he can't! *Alice has to tell him.*

There's no quantum way for Alice to tell Bob anything, because Alice has already collapsed their entangled states.

The only remaining way for Alice to tell Bob what she measured is by using classical means. She can send the values of her two measurements via radio, or she can use a laser to bounce light off the moon, or she can send him a newspaper with the measurements printed somewhere. No matter how she chooses to send this information to Bob, she needs to use classical means, which are limited by the speed of light.

Bob Recovers $|\sigma\rangle$

Once Bob receives the two classical bits that Alice sent, telling him what she measured, he can recover $|\sigma\rangle$ from his qubit. His work is made easier by another remarkable feature of the structure of the teleportation state: Bob will need only two qugates!

The idea is that Bob will use the received classical bits as controls on a controlled-X qugate and a controlled-Z qugate. We've only discussed using quantum bits as controls, but we can use classical bits as well. You can think of the classical bits 0 and 1 as the qubits $|0\rangle$ and $|1\rangle$.

To see what Alice's measurements tell Bob, I've repeated the teleportation state $|\tau\rangle$ from Equation 7.1 here as Equation 7.14.

$$|\tau\rangle = \frac{1}{2}\Big(|00\rangle I\,|\sigma\rangle + |01\rangle X\,|\sigma\rangle + |10\rangle Z\,|\sigma\rangle + |11\rangle XZ\,|\sigma\rangle \Big) \tag{7.14}$$

Bob's actions to recover $|\sigma\rangle$ are shown in Figure 7-8.

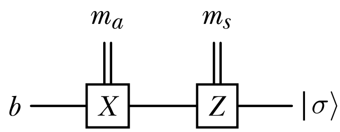


Figure 7-8: Bob decoding his qubit based on Alice's classical bits

Let's look at the four possible pairs of bits that Bob might receive.

If Alice measured 00, then $m_a = m_s = 0$. Equation 7.14 tells us that when Alice measures 00, Bob has $I\,|\sigma\rangle = |\sigma\rangle$. Since both controls are 0, neither qugate is applied. This is just right, because Bob already has $I\,|\sigma\rangle = |\sigma\rangle$, and he's done.

If Alice measured 01, then $m_s = 0$ and $m_a = 1$. Equation 7.14 tells us that Bob's qubit is in state $X\,|\sigma\rangle$. Bob applies the controlled-X qugate, and because X is its own inverse, this gives him $XX\,|\sigma\rangle = |\sigma\rangle$, and he's recovered $|\sigma\rangle$.

If Alice measured 10, then $m_s = 1$ and $m_a = 0$. This tells Bob to apply the controlled-Z qugate. Like the X qugate, Z is its own inverse, so Bob gets $ZZ\,|\sigma\rangle = |\sigma\rangle$.

Finally, If Alice measured 11, then $m_a = m_s = 1$, and Bob knows that his qubit is $XZ\,|\sigma\rangle$. To undo the transformation XZ, Bob applies X and then Z in that order, which is equivalent to applying the single operator ZX (remember to read the operators from right to left). The steps are shown in Equation 7.15.

$$\begin{aligned} ZX(XZ)\,|\sigma\rangle &= Z(XX)Z\,|\sigma\rangle && \text{Regroup matrix multiplies} \\ &= ZZ\,|\sigma\rangle && \text{Since } XX = I \\ &= |\sigma\rangle && \text{Since } ZZ = I \end{aligned} \tag{7.15}$$

It's important to keep the order of the operations at each point in the algorithm clear in your mind. If Alice's qubit is in the state $XZ\,|\sigma\rangle$, then to recover $|\sigma\rangle$ Bob has to apply the inverse of operator XZ, which is ZX.

And we're done. For each of Alice's four possible measurements, Bob has successfully ended up with his qubit in the state $|\sigma\rangle$. We've teleported $|\sigma\rangle$ from Alice to Bob!

Drawing the Teleportation Protocol

Let's now put all the pieces together, giving us Figure 7-9.

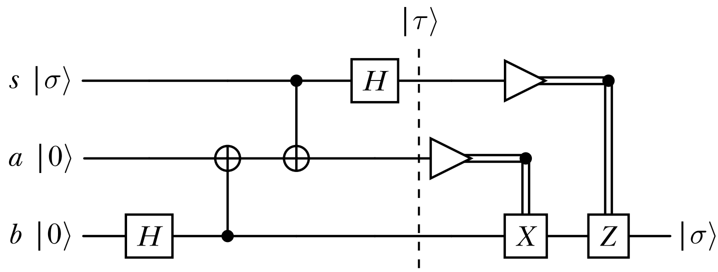


Figure 7-9: The full teleportation circuit as it's normally drawn, with the teleportation state $|\tau\rangle$ marked

Figure 7-9 is the complete, traditional quantum teleportation protocol. This theoretical process has been experimentally confirmed [21].

However, Figure 7-9 may be somewhat misleading, because it shows all three qubits at the far left, suggesting that Alice and Bob have them all in their control at the start of the process. But if Alice already has s in the state $|\sigma\rangle$, and Bob is standing there, she could just hand the s qubit (protected carefully) to Bob, and there would be no need for teleporting anything!

For that reason, I prefer drawing this as in the overall recap of Figure 7-10. The delayed introduction of s clarifies that s isn't yet in the state $|\sigma\rangle$ when Alice and Bob are entangling a and b. Only later does Alice compute s, and then continue the protocol.

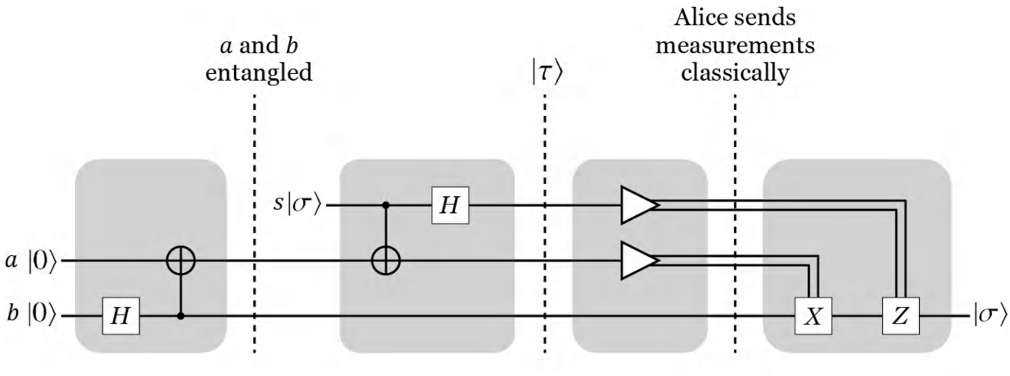


Figure 7-10: A recap of the quantum teleportation algorithm

Teleportation is usually drawn as in Figure 7-9, so keep in mind that in practice, the qubit s is usually not in the state $|\sigma\rangle$ at the start, when Alice and Bob are creating their entangled pair.

Probabilistic Teleportation

We've seen that after Alice has measured her qubits, Bob's qubit is in one of the four states in the superposition $|\tau\rangle$, but he doesn't know which one. Alice has to tell him by sending him two classical bits by classical means.

But let's suppose that for some reason, Alice can't send Bob her bits. Is the situation hopeless for Bob, or is there some way, perhaps with a combination of effort and luck, that he will be able to recover $|\sigma\rangle$ from his qubit?

Let's try another thought experiment.

Suppose that there's been a terrible accident on Mars. The habitat blew up when Bob was out on a mission, leaving Bob the only survivor. The explosion also damaged the rocket he and his colleagues were going to use to return to Earth, and almost all the fuel has leaked out. There's enough fuel to lift off the surface, but there isn't nearly enough to get the rocket back to Earth.

Bob's supplies will run out long before a rescue mission can reach him, so he needs to find a new way home.

Luckily, an earlier Mars mission placed an emergency rescue satellite in Mars orbit. If Bob can reach it, the resources there will not only keep him alive, but he'll be able to repair his rocket. And there's enough fuel there to fill the rocket's tanks. It's a plan to get home!

Getting his damaged rocket safely up to the rescue satellite will require an elaborate flight plan with multiple steps that he'll have to perform at the right moments. The specific plan for any given day will depend a lot on the local weather.

Unfortunately, Bob doesn't have access to the weather satellites above Mars. But he does have a working radio, and he contacts Alice, who can read the weather satellite data without a problem. They agree that Alice will use that data to work out a flight plan for the next day and send it to Bob. Because these flight plans are complicated, she'll encode the entire plan into a single state, $|\sigma\rangle$. This is a good strategy for them, because before Bob left, he and Alice created lots and lots of entangled pairs to use for teleporting states over the duration of Bob's mission. Bob's half of each pair survived the accident. So once Alice tells him her measurements, Bob can take down the bottle containing the next qubit to be used and apply Alice's bits to put it into the state $|\sigma\rangle$.

On Mars, Bob has cobbled together a decoder to turn a quantum state sent by Alice into a flight plan. He's also written a simulation program that will look at a flight plan and tell him whether it's safe and he'll reach the satellite, or it's unsafe and the rocket will blow up, along with Bob.

The next morning, as Bob prepares to hear from Alice what her measurements were, Bob's radio won't even turn on. It's busted, and he doesn't have the parts to fix it, so now he's lost all touch with Alice. He can't get her measurements, so he can't confidently process his qubit to turn it into $|\sigma\rangle$.

Bob isn't completely without hope, though. He can just plain old guess. Suppose he guesses that Alice measured 00, so his qubit is in the state $I\,|\sigma\rangle$, and he doesn't have to process it. He has a three in four chance of being wrong, but more optimistically, a one in four chance of being right!

So he feeds his qubit into the decoder, which gives him a flight plan. He then gives that to the safety testing program. If he's lucky, the test will tell him that he guessed correctly and that the flight plan makes sense and is safe. But if he's unlucky, then his guess was wrong (that is, his qubit wasn't in the state $|\sigma\rangle$, but one of the other states in $|\tau\rangle$). In that case, the decoder will have produced a nonsensical flight plan, and the test will tell him that following that plan would end in disaster.

If Alice and Bob were sharing only a single entangled pair, this would be the end of the story. Bob would have no option but to launch the rocket anyway and hope for the best.

Is there some way, *any way*, that Bob can improve his odds?

Recall that Alice and Bob created not one entangled pair before Bob left, but many dozens or hundreds of them. They intended to use them to teleport different quantum states over the course of Bob's mission. But now that Alice knows Bob's radio is out, she'll use them all right away. Alice will run the quantum protocol to teleport the same state $|\sigma\rangle$ over every pair of entangled qubits. Since she can't clone the $|\sigma\rangle$ she's made, she runs her plan-making program many times, creating many distinct qubits that are all $|\sigma\rangle$. She plugs each of these qubits into the teleportation protocol, and even measures the output bits, though she can't send them to Bob. At this point, she's done all she can.

Back on Mars, suppose that Bob's guess for his first qubit resulted in a meaningless and unsafe flight plan. He hopes that Alice is following their backup plan, and gives her some time to compute and entangle $|\sigma\rangle$ on all of their remaining pairs.

After a little while, he'll take down the next of his entangled qubits, guess again, and process the qubit according to his guess of what state the qubit is in (he could just guess it's $|\sigma\rangle$ every time, and apply no qugates to it). Then Bob will decode his qubit and test the resulting flight plan, hoping it will be safe.

The process is shown graphically in Figure 7-11. Alice measures s and a, producing classical bits m_s and m_a, but she does nothing with them. The measurements were just to collapse the states of the qubits. Bob then feeds his qubit b into the decoding algorithm G that turns that qubit into a flight path, represented by the binary number g that comes from measuring the output of G. Because g is a classical binary number, Bob can make as many copies of it as he pleases.

So, Bob makes a copy of g and feeds it into his test, along with whatever other inputs it needs. If the test says the flight plan is safe, then he can follow the steps in g and he's all set for launch!

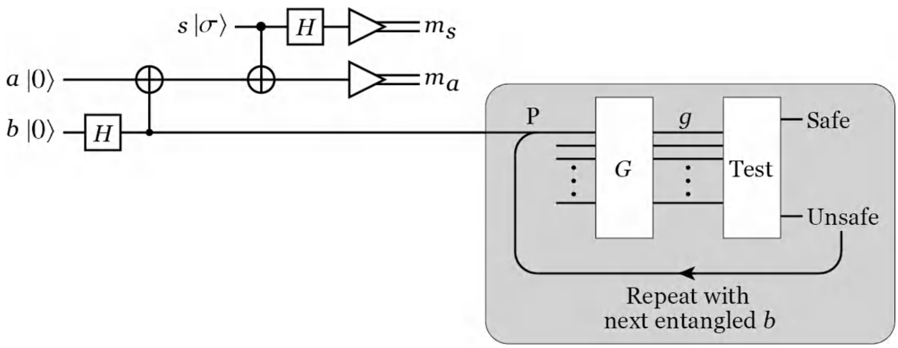


Figure 7-11: After Alice has measured her qubits, Bob guesses that he has $|\sigma\rangle$, computes the binary bitstring g, and then tests that bitstring to see if it's a safe flight plan.

If the test tells him that the flight plan is unsafe, he takes down his next qubit b, makes another guess, and tries again.

Because there's no way for Bob to be sure beforehand that a given qubit actually has the value he's guessing, we might call this process *probabilistic teleportation*.

An important thing to keep in mind is that Bob doesn't really care about the state $|\sigma\rangle$. He'll *use* each qubit, rather than study it, and it's the results of the decoder and test that he cares about.

Once Bob has guessed at the state of any qubit b, he has a one in four chance, or a probability of 0.25, that he'll have guessed correctly and the test will tell him the plan is safe. Those aren't great odds.

What are the chances that when Bob uses this approach, he will ultimately guess right, and thereby get a safe flight plan that could save his life?

To see Bob's chances of success, consider his odds of failure. After one guess (and any processing it might require), there's a three in four chance that b is *not* in the state $|\sigma\rangle$. Thus, Bob has a 0.75 probability of being wrong (and getting an unsafe plan). But this means he has a $1 - 0.75 = 0.25$ probability of being right (and getting a safe plan). After two repeats of the teleportation, his probability of guessing incorrectly both times is $0.75 \times 0.75 = 0.75^2 = 0.5625$. Thus, his probability of being correct at least once is $1 - 0.5625 = 0.4375$. Much better!

After n repeats of the protocol, the probability that Bob has guessed correctly at least once is $1 - 0.75^n$, which I've graphed in Figure 7-12.

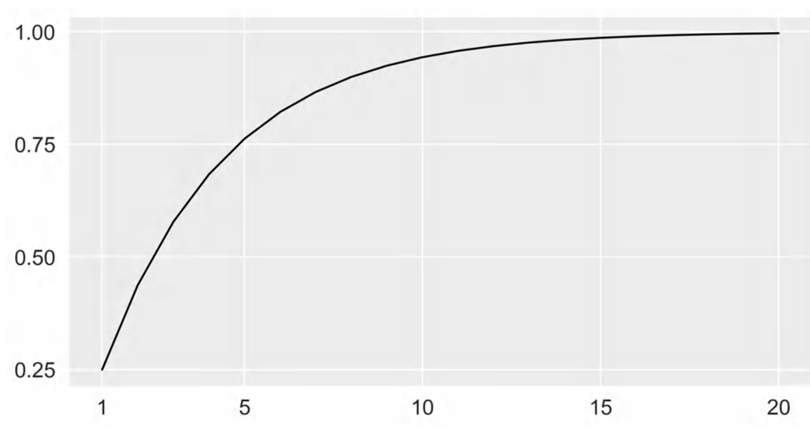


Figure 7-12: A plot of $1 - 0.75^n$ *for* n *from* 1 *to* 20

After 10 repeats, Bob has about a 0.94 probability of having been right at least once. After 20 repeats, his probability of having been right at least once is almost 0.997.

If Alice and Bob are willing to share 20 entangled qubits before Bob leaves, *and* Alice computes her side of the protocol from scratch 20 times, *and* Bob runs his decoder and test 20 times, there's about a 99.7 percent likelihood that Bob will have guessed correctly at some point and obtained a safe flight plan. He only needs one safe flight plan, and he can stop as soon as he has it.

After 50 attempts, Bob's chance of never guessing right, even once, is less than 1 in a million. But even though the odds of Bob guessing correctly go up with each repeat, if he's super unlucky, he might never guess correctly. Worse, Bob has no way to tell Alice about this failure.

As this example illustrates, when Alice and Bob are able or willing to share two classical bits, they can run the classical protocol once and teleport the qubit for sure. If they don't want to share those classical bits, or they're unable to, they can hope to teleport the qubit, but they'll have to put in a lot of extra effort, and they might still fail.

To be guaranteed success, Alice needs to send the two classical bits representing her measurements to Bob, using classical means. If they can't do that, they can hope that luck is on Bob's side and he'll get an answer that passes his test before he runs out of qubits.

This is only one of many interesting modifications to the basic teleportation algorithm [4] [157] [289]. You can read up on the references, or try your own ideas. Exploring variations on circuits you already understand is a great way to gain experience with quantum algorithms.

Summary

The teleportation protocol lets us transfer a quantum state from one qubit to another, which can be arbitrarily far away. The process requires that Alice and Bob already share an entangled pair of qubits, and that Alice can transmit two classical bits to Bob. Alice's measurements cause qubit s to collapse. This means that there is never more than one qubit in the state $|\sigma\rangle$.

The big surprise of teleportation is that the state we've transferred contains two complex numbers, each of which is built from two real numbers, for a total of four real numbers. These numbers can require arbitrary numbers of digits if written out, but they will be transferred with perfect precision.

Once Bob has operated on his state, he can measure it. As always, measurement will give him only a 0 or a 1, so there's no way to extract those four real numbers that were transmitted. But before measurement, Bob can use his qubit, now in state $|\sigma\rangle$, in further computations. So Bob can then build on Alice's work, using her result as an input to his own algorithm.

Although the collapse of Bob's qubit is immediate after Alice's measurements, the need to then share classical bits prevents us from using this protocol to send information faster than the speed of light. That's too bad, but it doesn't change the fact that quantum teleportation is still a pretty amazing feat.

If Alice and Bob have lost all classical communication, and they have some additional resources, they can use a probabilistic approach that is likely to give Bob the state $|\sigma\rangle$ eventually. But he'll probably have to try several different saved instances of b (independently computed by Alice) and then test each result.

In this chapter, we looked at the matrix elements behind Alice's computation of the teleportation state $|\tau\rangle$, and then we unpacked that state to teleport $|\sigma\rangle$ to Bob. We had to deal with a few big matrices, but they were manageable. In general, a system of n qubits will need an operator described by a matrix 2^n elements on a side, which quickly becomes too big to write out and manually compute with. So from now on, we'll focus on the algebraic approach most of the time, rather than writing out the matrices and kets.

Teleportation is an amazing algorithm, and it shows the power of entanglement for sharing information at a great distance. It's pretty incredible that Alice can perform some quantum operations on qubits on Earth and change the states of Bob's qubits on Mars (or even in another galaxy far, far away).

In the next few chapters, we'll look at more quantum algorithms, each of which introduces a new concept into our quantum repertoire.