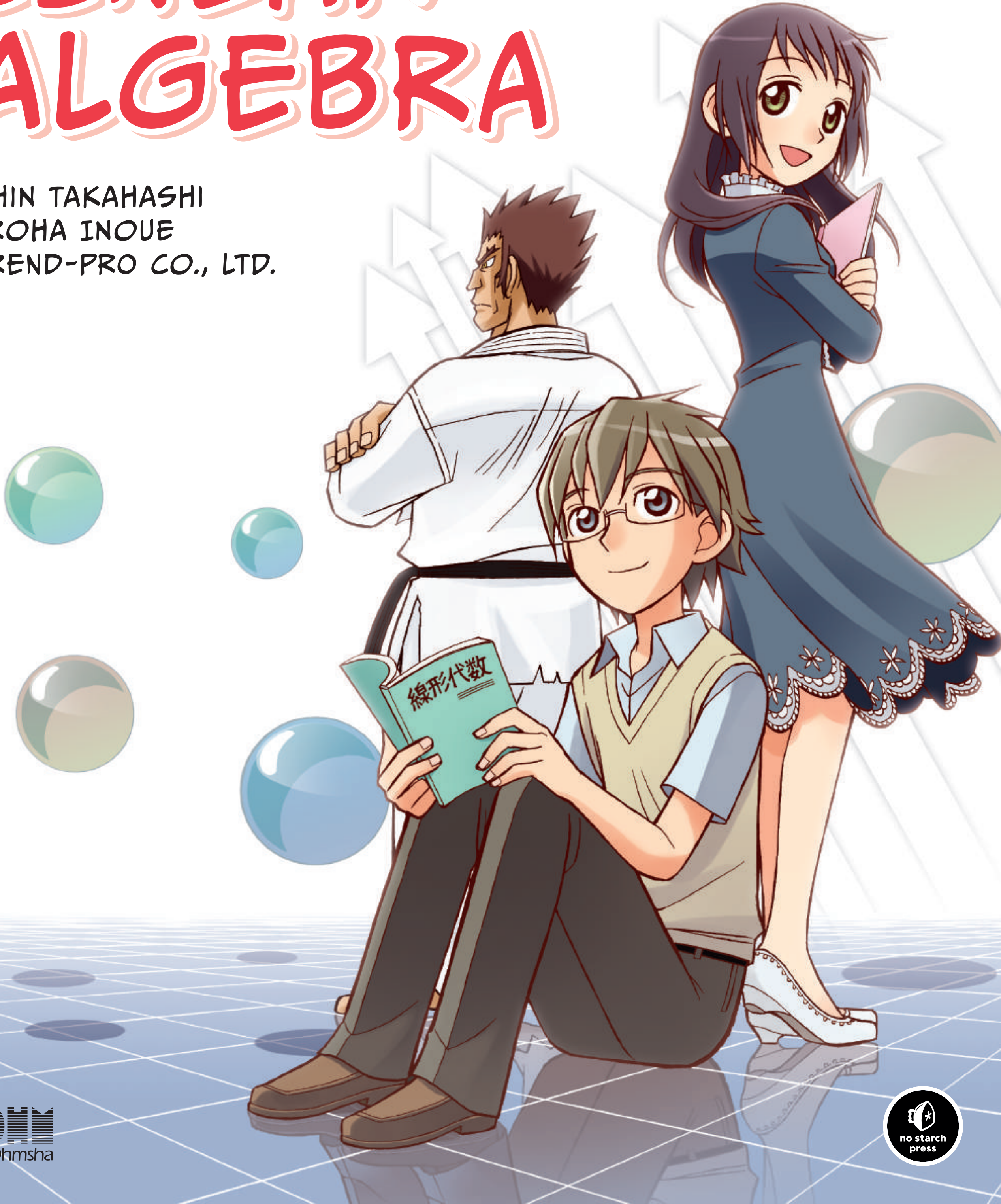
THE MANGA GUIDE™ TO

COMICS INSIDE!

LINEAR ALGEBRA

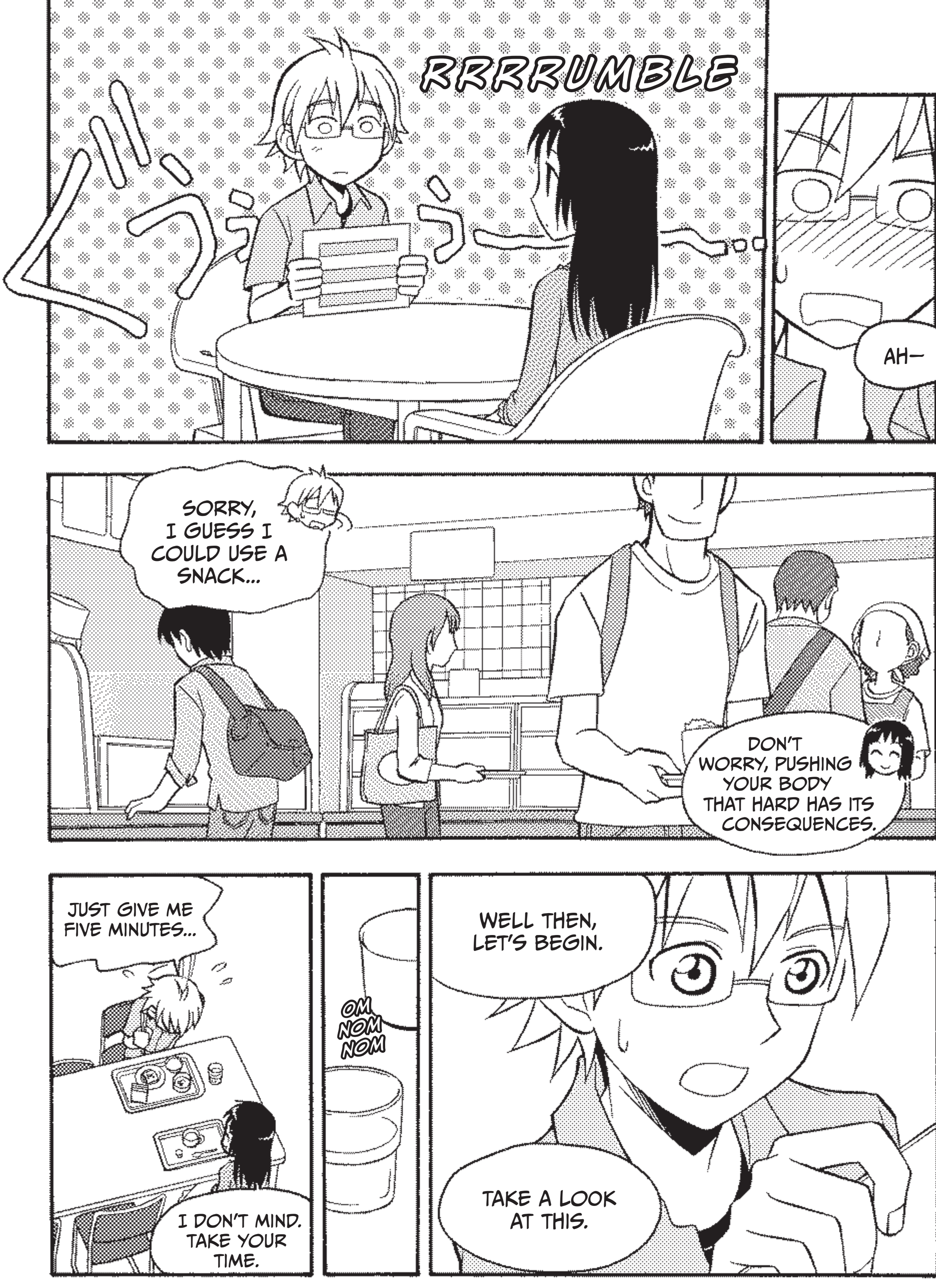
SHIN TAKAHASHI
IROHA INOUE
TREND-PRO CO., LTD.

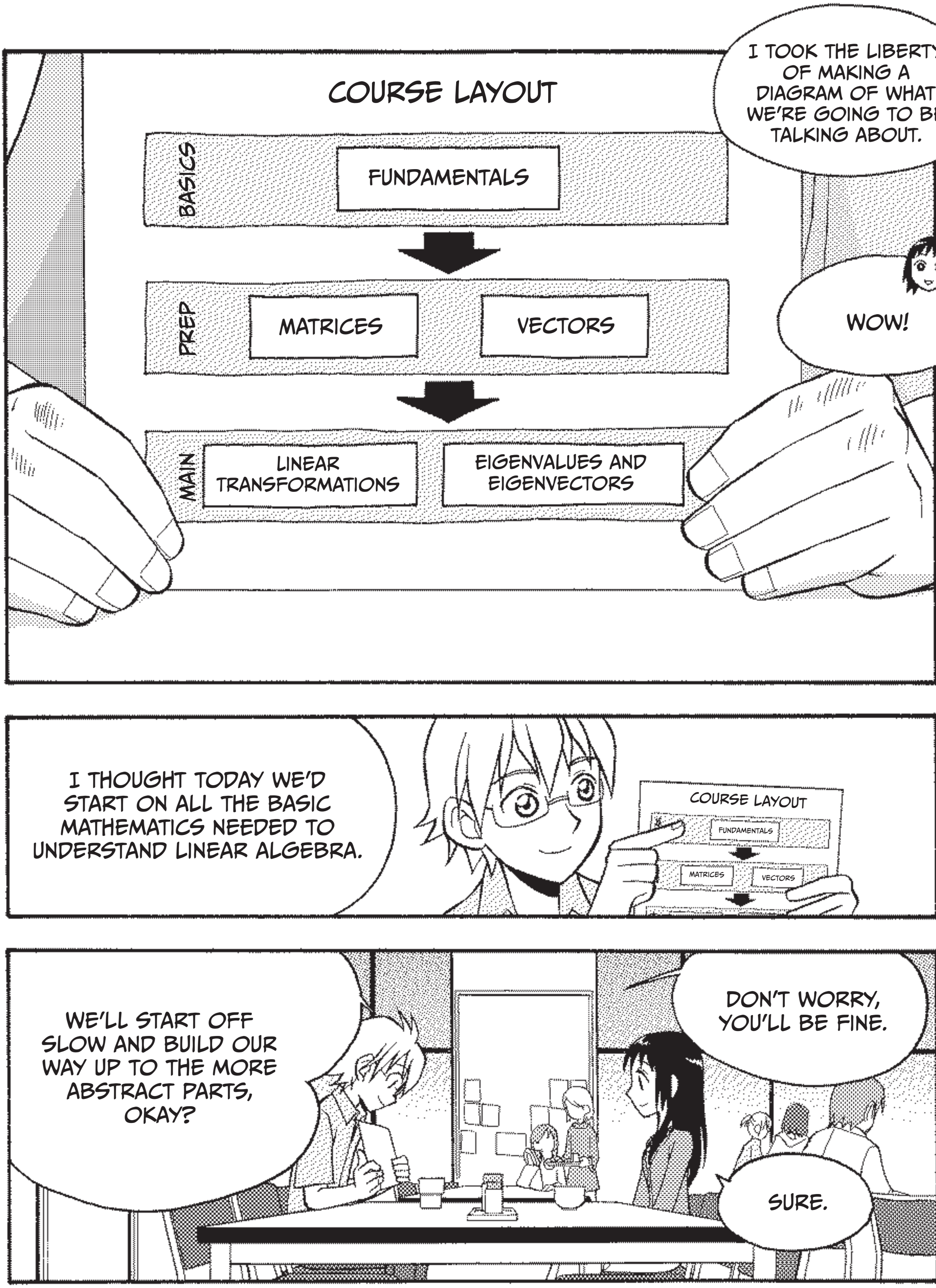


Ohmsha
no starch press

2
THE FUNDAMENTALS


97
98
99
YOU'VE GOT TO KNOW YOUR BASICS!
GGHH
WHUMP
100...
D-DONE!
YOU WISH! AFTER YOU'RE DONE WITH THE PUSHUPS, I WANT YOU TO START ON YOUR LEGS! THAT MEANS SQUATS! GO GO GO!
HEY... I THOUGHT WE'D START OFF WITH...
REIJI, YOU SEEM PRETTY OUT OF IT TODAY.
ARE YOU OKAY?
I-I'LL BE FINE. TAKE A LOOK AT THI—


RRRRUMBLE
AH—
SORRY, I GUESS I COULD USE A SNACK...
DON'T WORRY, PUSHING YOUR BODY THAT HARD HAS ITS CONSEQUENCES.
JUST GIVE ME FIVE MINUTES...
I DON'T MIND. TAKE YOUR TIME.
OM NOM NOM
WELL THEN, LET'S BEGIN.
TAKE A LOOK AT THIS.


I TOOK THE LIBERTY OF MAKING A DIAGRAM OF WHAT WE'RE GOING TO BE TALKING ABOUT.
COURSE LAYOUT
BASICS
FUNDAMENTALS
PREP
MATRICES
VECTORS
MAIN
LINEAR TRANSFORMATIONS
EIGENVALUES AND EIGENVECTORS
WOW!
I THOUGHT TODAY WE'D START ON ALL THE BASIC MATHEMATICS NEEDED TO UNDERSTAND LINEAR ALGEBRA.
COURSE LAYOUT
FUNDAMENTALS
MATRICES
VECTORS
WE'LL START OFF SLOW AND BUILD OUR WAY UP TO THE MORE ABSTRACT PARTS, OKAY?
DON'T WORRY, YOU'LL BE FINE.
SURE.

COMPLEX NUMBERS

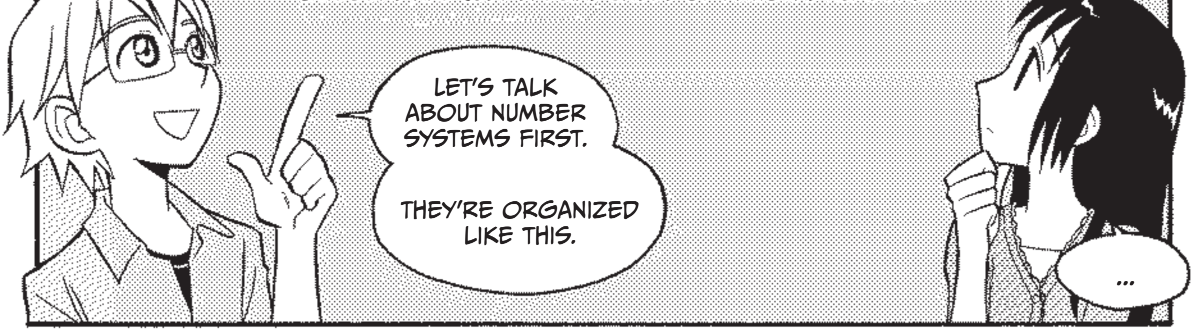
Complex numbers are written in the form

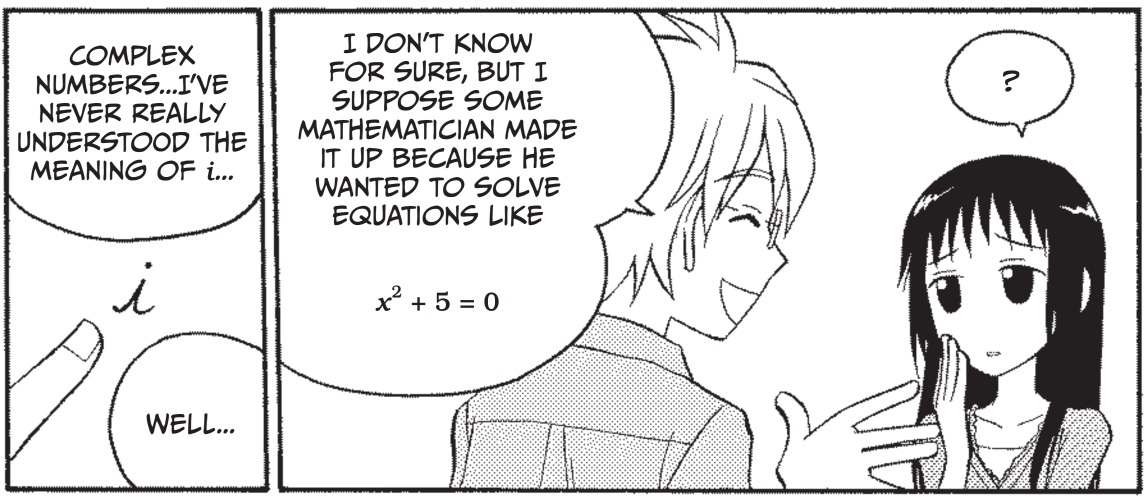
$$a + b \cdot i$$

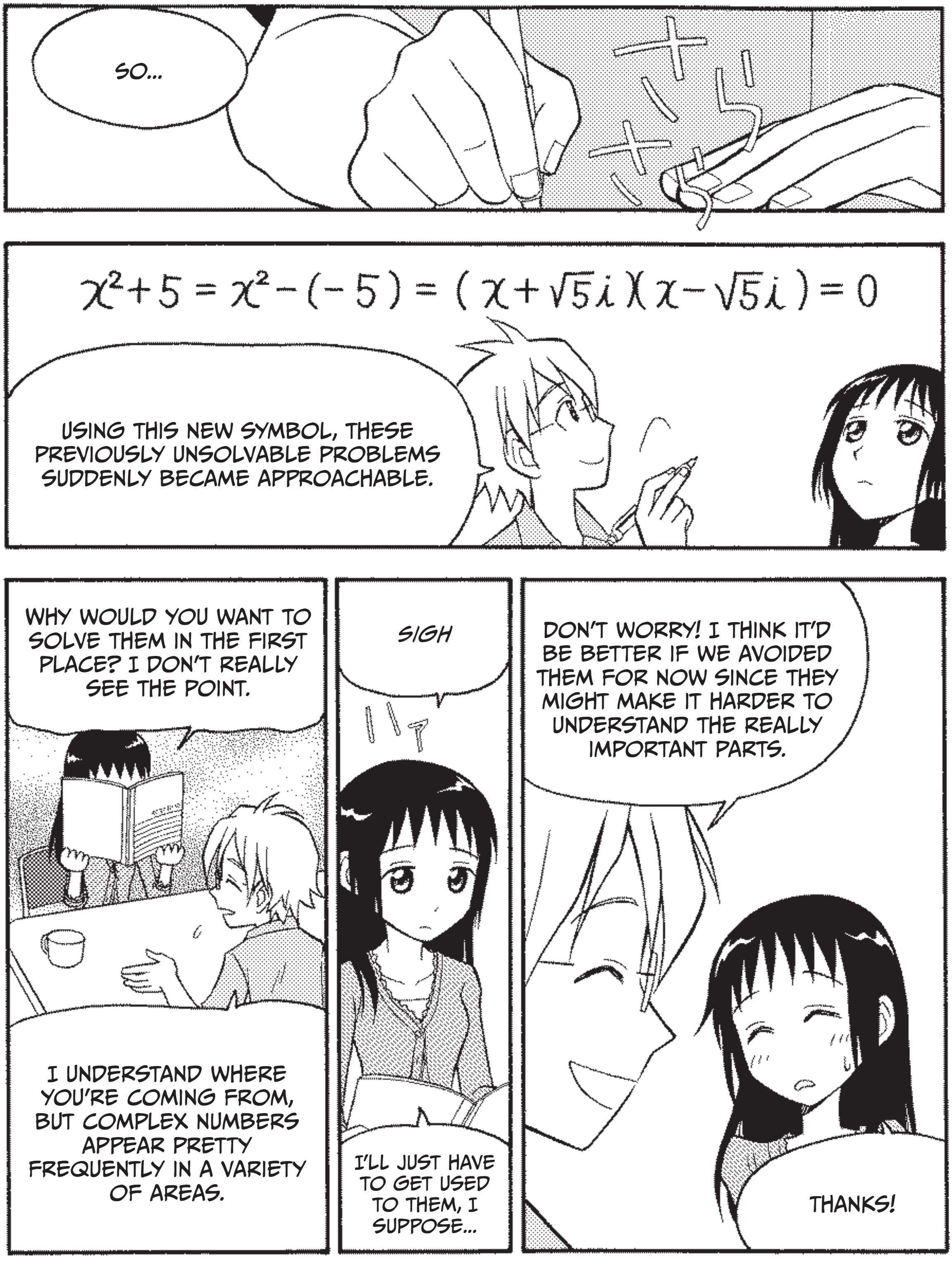
where a and b are real numbers and i is the *imaginary unit*, defined as $i = \sqrt{-1}$.

REAL NUMBERS			IMAGINARY NUMBERS
INTEGERS	RATIONAL NUMBERS* (NOT INTEGERS)	IRRATIONAL NUMBERS	
• Positive natural numbers • 0 • Negative natural numbers	• Terminating decimal numbers like 0.3 • Non-terminating decimal numbers like 0.333...	• Numbers like π and $\sqrt{2}$ whose decimals do not follow a pattern and repeat forever	• Complex numbers without a real component, like $0 + bi$, where b is a nonzero real number

* NUMBERS THAT CAN BE EXPRESSED IN THE FORM q / p (WHERE q AND p ARE INTEGERS, AND p IS NOT EQUAL TO ZERO) ARE KNOWN AS *RATIONAL NUMBERS*. INTEGERS ARE JUST SPECIAL CASES OF RATIONAL NUMBERS.

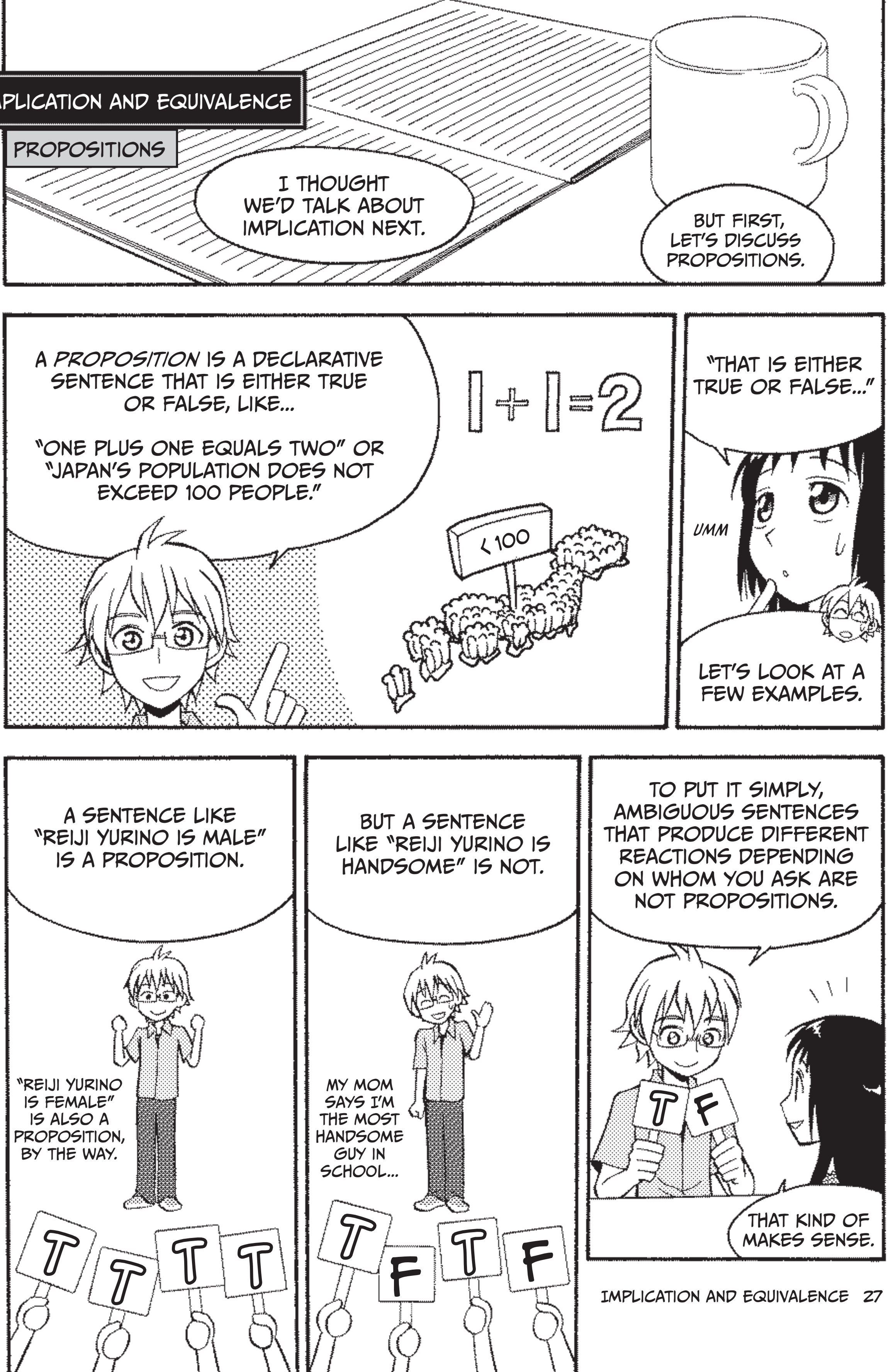


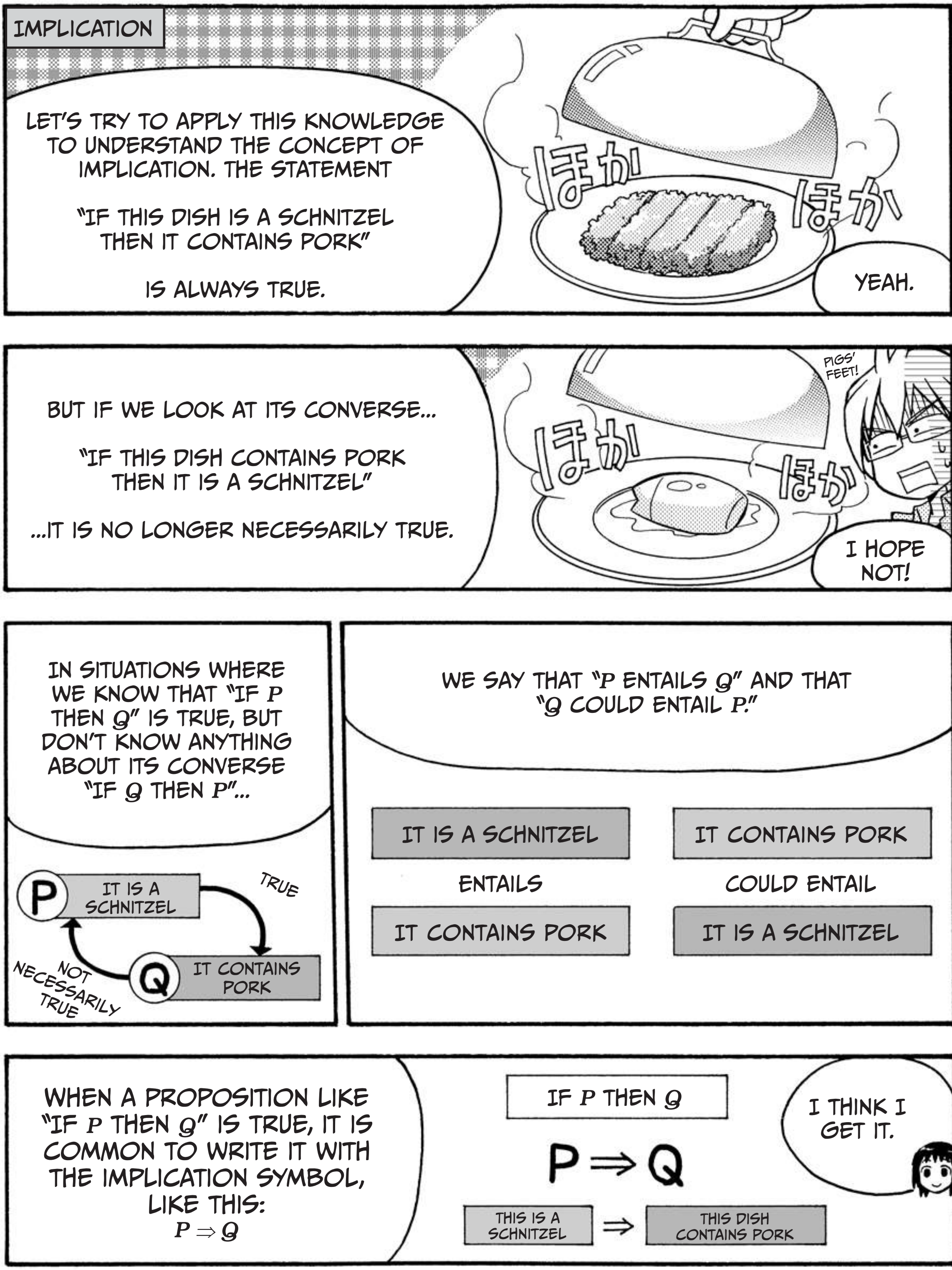


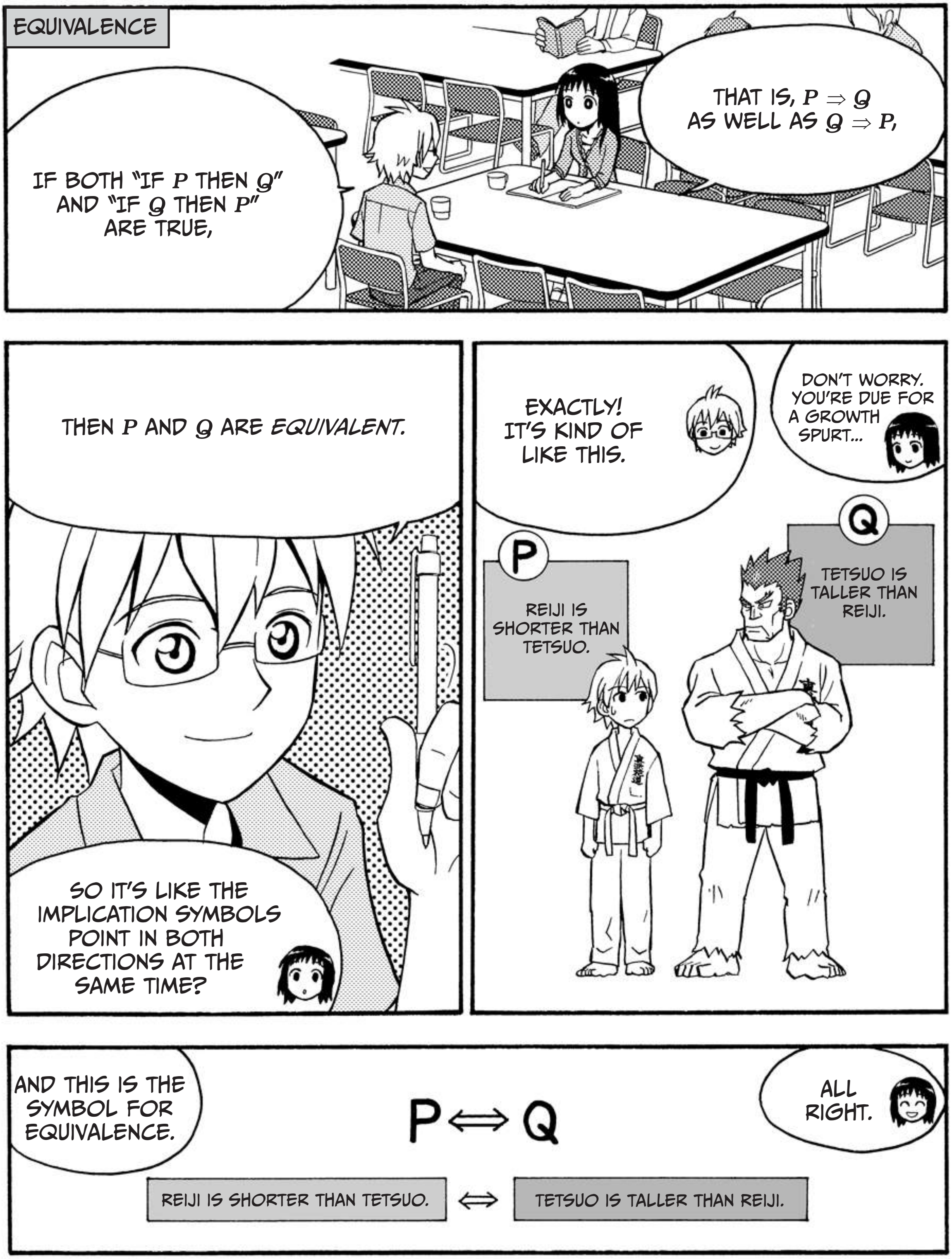

SO...
$x^2+5 = x^2-(-5) = (x+\sqrt{5}i)(x-\sqrt{5}i) = 0$
USING THIS NEW SYMBOL, THESE PREVIOUSLY UNSOLVABLE PROBLEMS SUDDENLY BECAME APPROACHABLE.
WHY WOULD YOU WANT TO SOLVE THEM IN THE FIRST PLACE? I DON'T REALLY SEE THE POINT.
I UNDERSTAND WHERE YOU'RE COMING FROM, BUT COMPLEX NUMBERS APPEAR PRETTY FREQUENTLY IN A VARIETY OF AREAS.
SIGH
I'LL JUST HAVE TO GET USED TO THEM, I SUPPOSE...
DON'T WORRY! I THINK IT'D BE BETTER IF WE AVOIDED THEM FOR NOW SINCE THEY MIGHT MAKE IT HARDER TO UNDERSTAND THE REALLY IMPORTANT PARTS.
THANKS!

IMPLICATION AND EQUIVALENCE

PROPOSITIONS

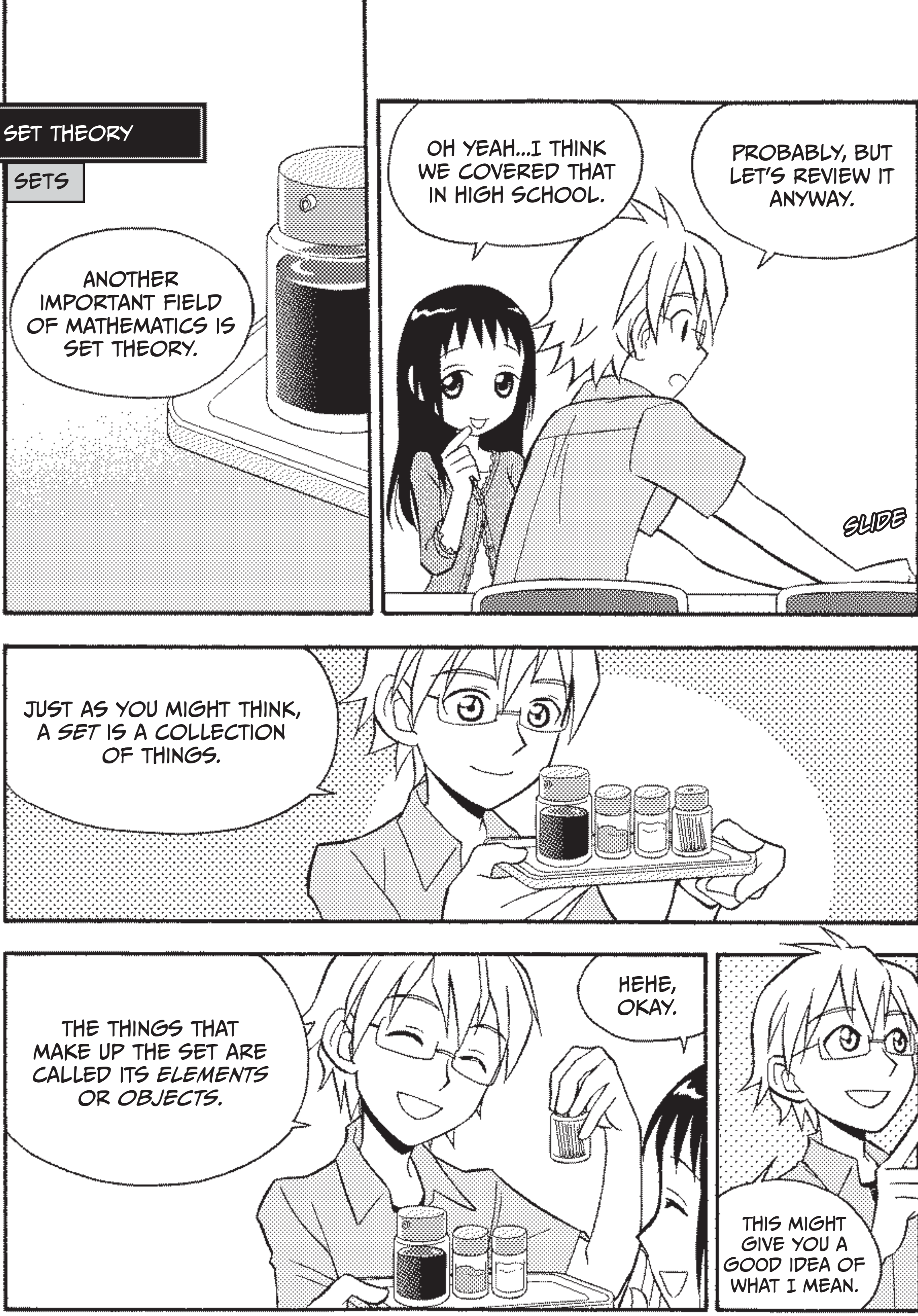



IMPLICATION
LET'S TRY TO APPLY THIS KNOWLEDGE TO UNDERSTAND THE CONCEPT OF IMPLICATION. THE STATEMENT
"IF THIS DISH IS A SCHNITZEL THEN IT CONTAINS PORK"
IS ALWAYS TRUE.
ほか
ほか
YEAH.
BUT IF WE LOOK AT ITS CONVERSE...
"IF THIS DISH CONTAINS PORK THEN IT IS A SCHNITZEL"
...IT IS NO LONGER NECESSARILY TRUE.
ほか
ほか
PIGS' FEET!
I HOPE NOT!
IN SITUATIONS WHERE WE KNOW THAT "IF P THEN Q" IS TRUE, BUT DON'T KNOW ANYTHING ABOUT ITS CONVERSE "IF Q THEN P"...
P
IT IS A SCHNITZEL
TRUE
Q
IT CONTAINS PORK
NOT NECESSARILY TRUE
WE SAY THAT "P ENTAILS Q" AND THAT "Q COULD ENTAIL P."
IT IS A SCHNITZEL
ENTAILS
IT CONTAINS PORK
IT CONTAINS PORK
COULD ENTAIL
IT IS A SCHNITZEL
WHEN A PROPOSITION LIKE "IF P THEN Q" IS TRUE, IT IS COMMON TO WRITE IT WITH THE IMPLICATION SYMBOL, LIKE THIS: P ⇒ Q
IF P THEN Q
P ⇒ Q
THIS IS A SCHNITZEL
⇒
THIS DISH CONTAINS PORK
I THINK I GET IT.


EQUIVALENCE
IF BOTH "IF P THEN Q" AND "IF Q THEN P" ARE TRUE,
THAT IS, P ⇒ Q AS WELL AS Q ⇒ P,
THEN P AND Q ARE EQUIVALENT.
SO IT'S LIKE THE IMPLICATION SYMBOLS POINT IN BOTH DIRECTIONS AT THE SAME TIME?
EXACTLY! IT'S KIND OF LIKE THIS.
DON'T WORRY. YOU'RE DUE FOR A GROWTH SPURT...
P
REIJI IS SHORTER THAN TETSUO.
Q
TETSUO IS TALLER THAN REIJI.
AND THIS IS THE SYMBOL FOR EQUIVALENCE.
P ⇔ Q
ALL RIGHT.
REIJI IS SHORTER THAN TETSUO. ⇔ TETSUO IS TALLER THAN REIJI.

SET THEORY

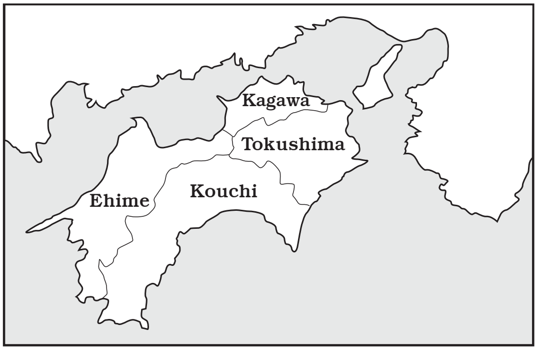
SETS



EXAMPLE 1

The set "Shikoku," which is the smallest of Japan's four islands, consists of these four elements:

- Kagawa-ken[1]
- Ehime-ken
- Kouchi-ken
- Tokushima-ken

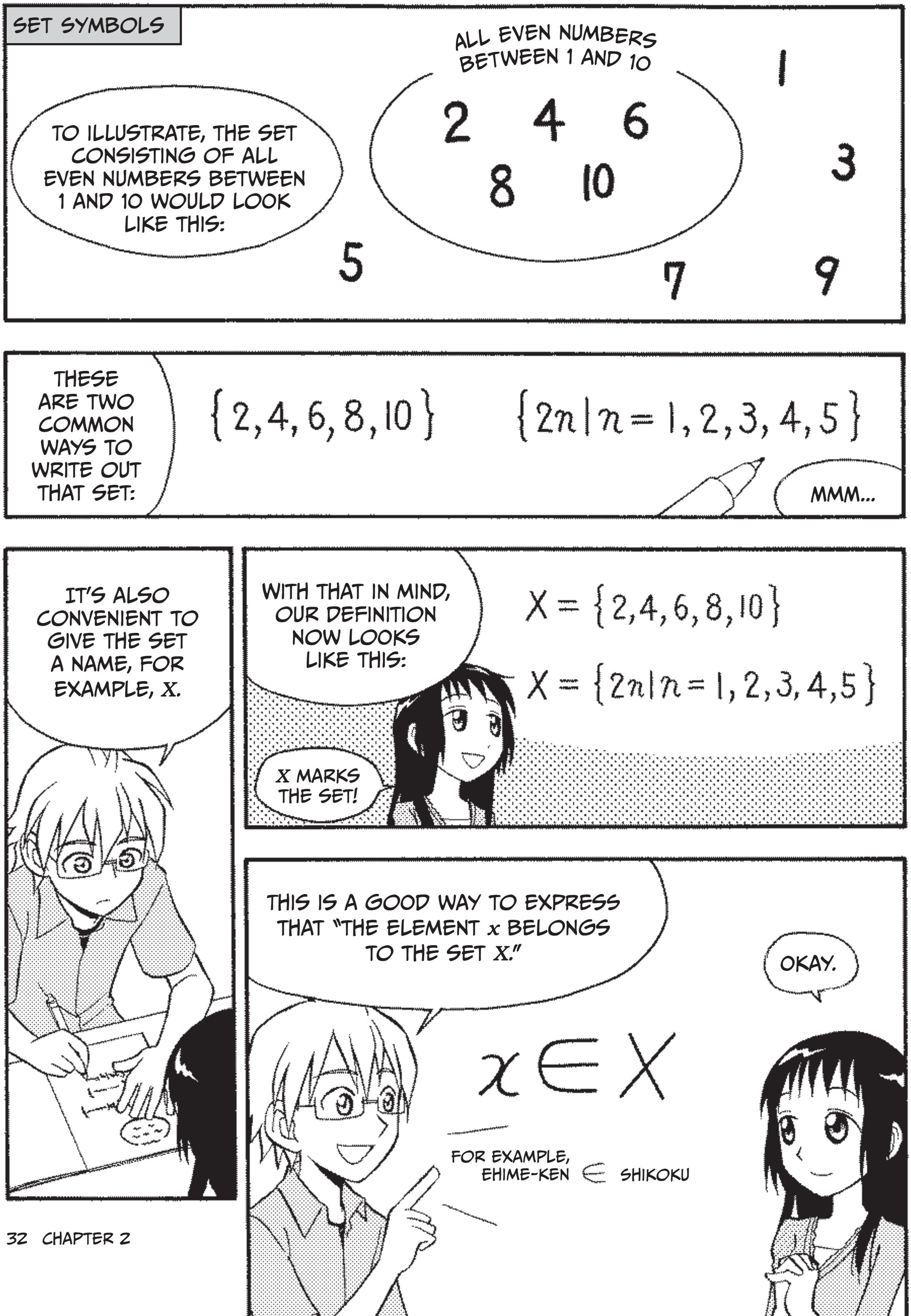


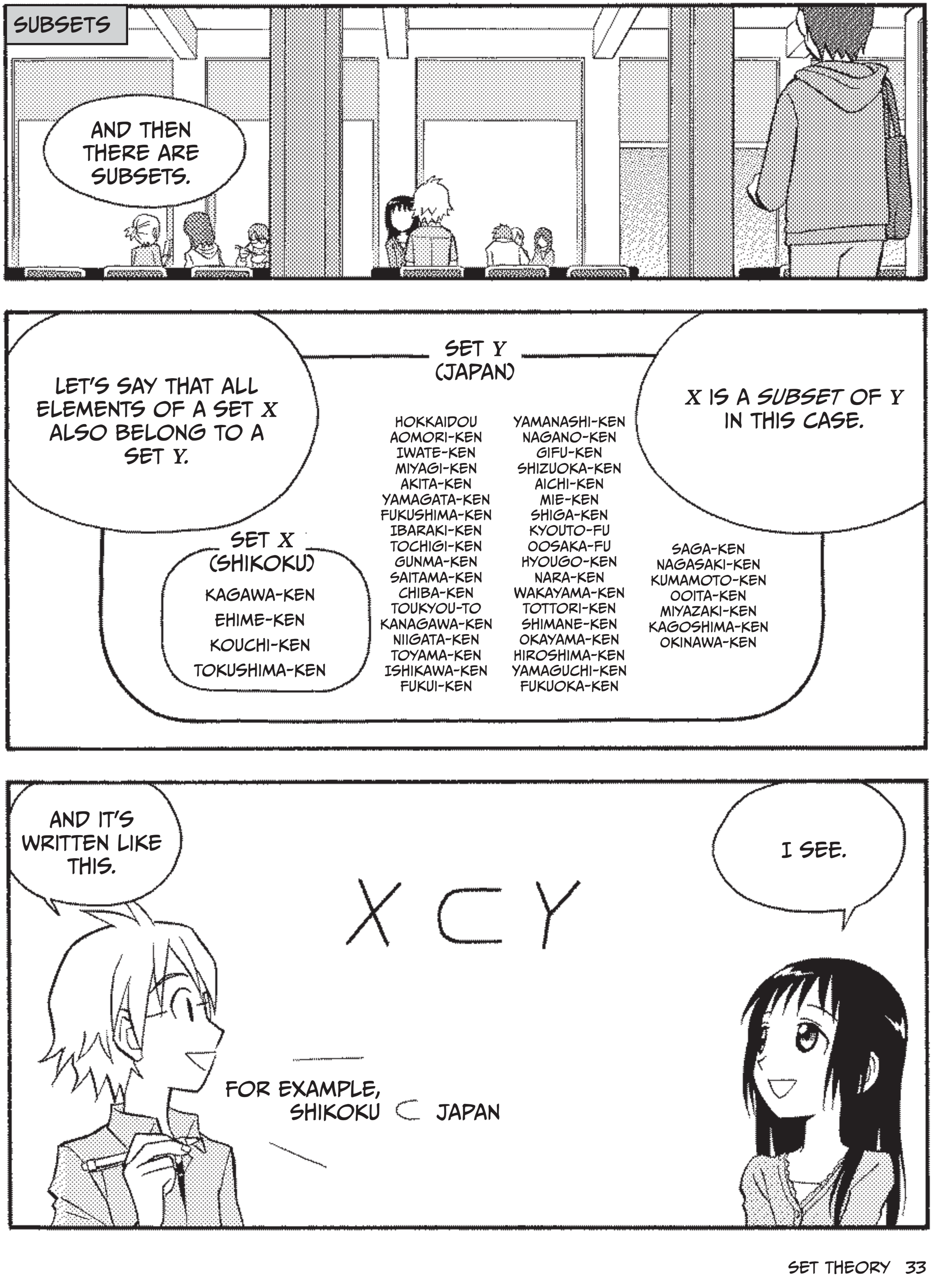
EXAMPLE 2

The set consisting of all even integers from 1 to 10 contains these five elements:

- 2
- 4
- 6
- 8
- 10

1. A Japanese *ken* is kind of like an American state.


SET SYMBOLS
TO ILLUSTRATE, THE SET CONSISTING OF ALL EVEN NUMBERS BETWEEN 1 AND 10 WOULD LOOK LIKE THIS:
ALL EVEN NUMBERS BETWEEN 1 AND 10
2 4 6 8 10
1 3 5 7 9
THESE ARE TWO COMMON WAYS TO WRITE OUT THAT SET:
$\{2, 4, 6, 8, 10\}$
$\{2n \mid n = 1, 2, 3, 4, 5\}$
MMM...
IT'S ALSO CONVENIENT TO GIVE THE SET A NAME, FOR EXAMPLE, X.
WITH THAT IN MIND, OUR DEFINITION NOW LOOKS LIKE THIS:
$X = \{2, 4, 6, 8, 10\}$
$X = \{2n \mid n = 1, 2, 3, 4, 5\}$
X MARKS THE SET!
THIS IS A GOOD WAY TO EXPRESS THAT "THE ELEMENT x BELONGS TO THE SET X."
OKAY.
$x \in X$
FOR EXAMPLE, EHIME-KEN $\in$ SHIKOKU

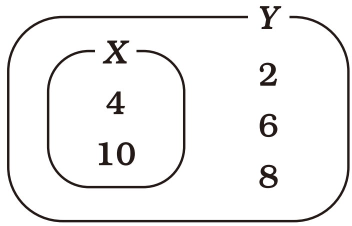

SUBSETS
AND THEN THERE ARE SUBSETS.
LET'S SAY THAT ALL ELEMENTS OF A SET X ALSO BELONG TO A SET Y.
SET Y (JAPAN)
HOKKAIDOU
AOMORI-KEN
IWATE-KEN
MIYAGI-KEN
AKITA-KEN
YAMAGATA-KEN
FUKUSHIMA-KEN
IBARAKI-KEN
TOCHIGI-KEN
GUNMA-KEN
SAITAMA-KEN
CHIBA-KEN
TOUKYOU-TO
KANAGAWA-KEN
NIIGATA-KEN
TOYAMA-KEN
ISHIKAWA-KEN
FUKUI-KEN
YAMANASHI-KEN
NAGANO-KEN
GIFU-KEN
SHIZUOKA-KEN
AICHI-KEN
MIE-KEN
SHIGA-KEN
KYOUTO-FU
OOSAKA-FU
HYOUGO-KEN
NARA-KEN
WAKAYAMA-KEN
TOTTORI-KEN
SHIMANE-KEN
OKAYAMA-KEN
HIROSHIMA-KEN
YAMAGUCHI-KEN
FUKUOKA-KEN
SAGA-KEN
NAGASAKI-KEN
KUMAMOTO-KEN
OOITA-KEN
MIYAZAKI-KEN
KAGOSHIMA-KEN
OKINAWA-KEN
SET X (SHIKOKU)
KAGAWA-KEN
EHIME-KEN
KOUCHI-KEN
TOKUSHIMA-KEN
X IS A SUBSET OF Y IN THIS CASE.
AND IT'S WRITTEN LIKE THIS.
X ⊂ Y
I SEE.
FOR EXAMPLE, SHIKOKU ⊂ JAPAN

EXAMPLE 1

Suppose we have two sets *X* and *Y*:

$X = \{ 4, 10 \}$
$Y = \{ 2, 4, 6, 8, 10 \}$

X is a subset of *Y*, since all elements in *X* also exist in *Y*.

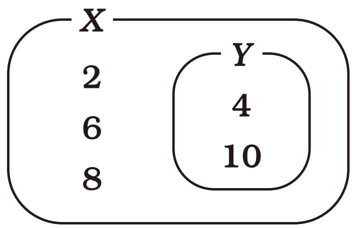


EXAMPLE 2

Suppose we switch the sets:

$X = \{ 2, 4, 6, 8, 10 \}$
$Y = \{ 4, 10 \}$

Since all elements in *X* don't exist in *Y*, *X* is no longer a subset of *Y*.

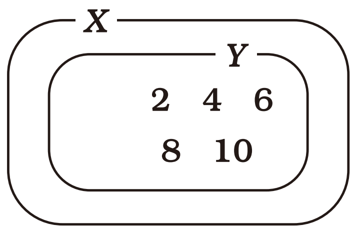


EXAMPLE 3

Suppose we have two equal sets instead:

$X = \{ 2, 4, 6, 8, 10 \}$
$Y = \{ 2, 4, 6, 8, 10 \}$

In this case, both sets are subsets of each other. So *X* is a subset of *Y*, and *Y* is a subset of *X*.

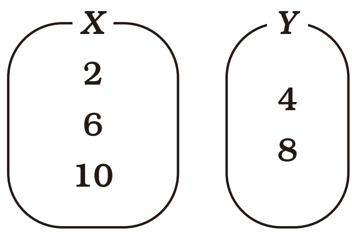


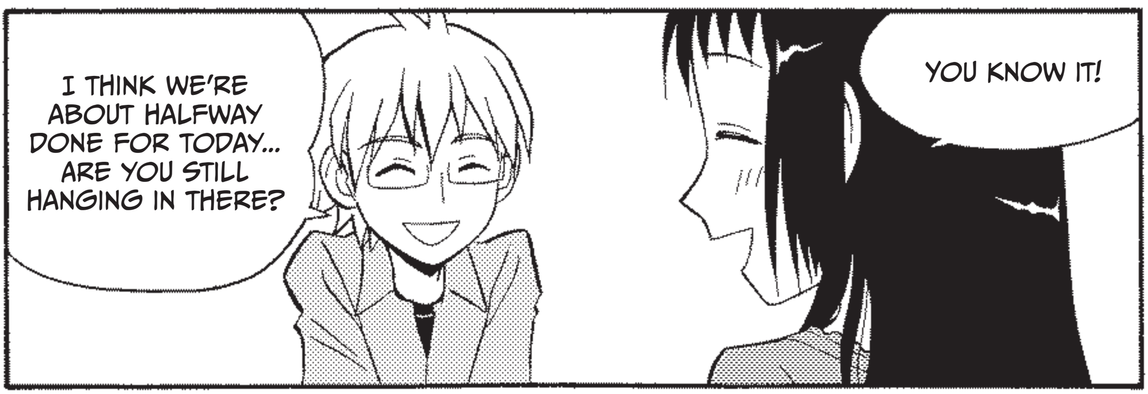
EXAMPLE 4

Suppose we have the two following sets:

$X = \{ 2, 6, 10 \}$
$Y = \{ 4, 8 \}$

In this case neither *X* nor *Y* is a subset of the other.

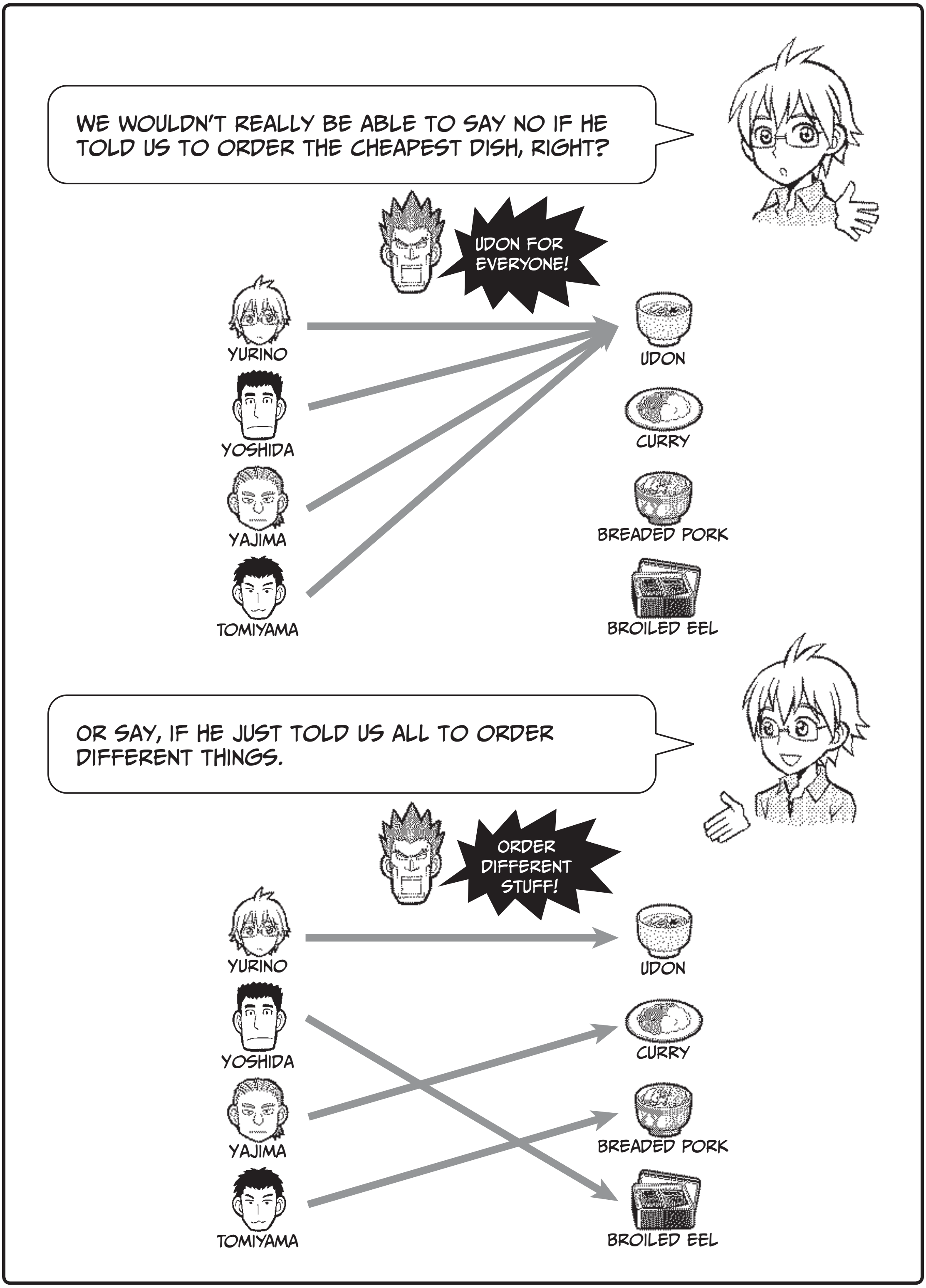


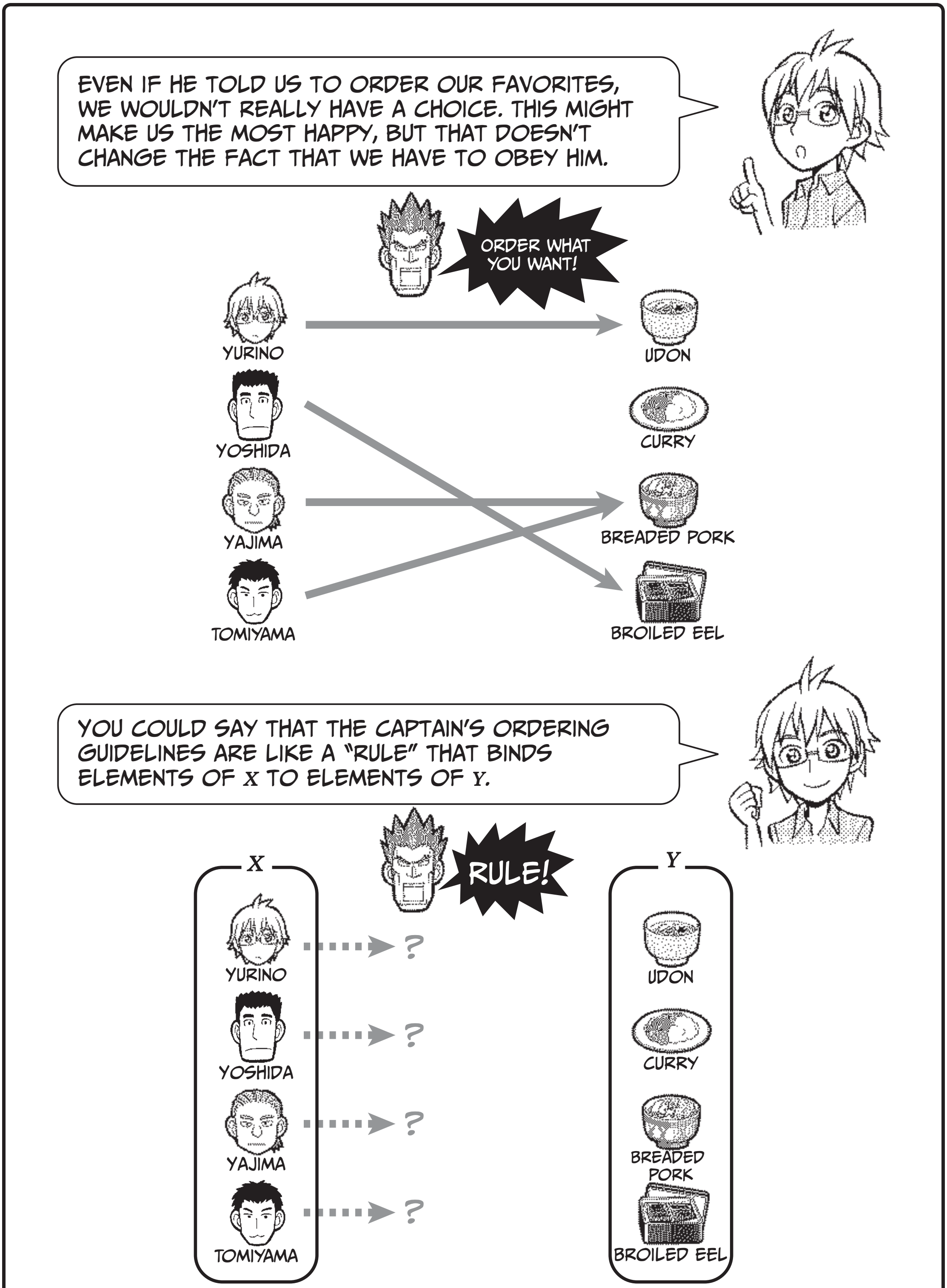


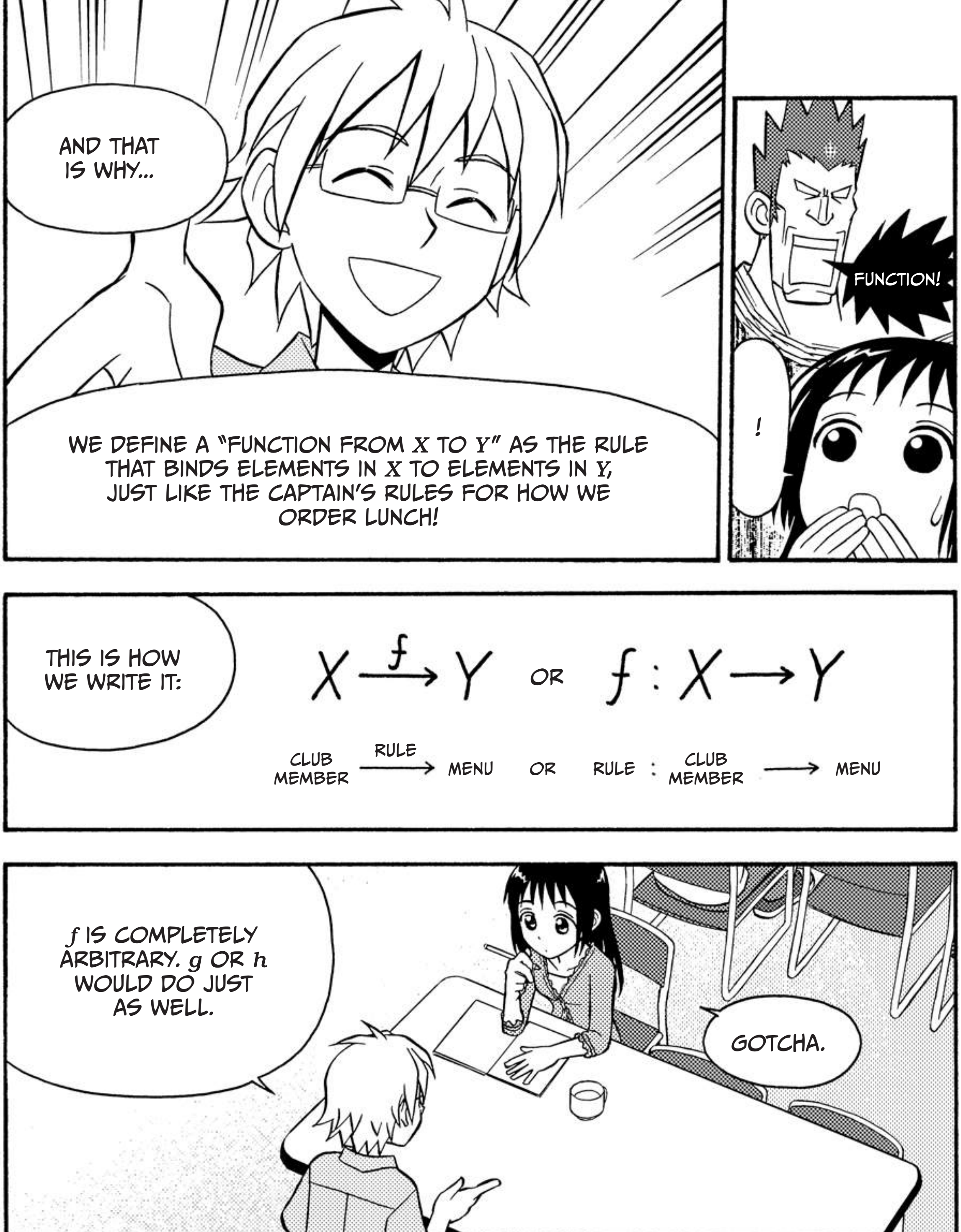
FUNCTIONS


I THOUGHT WE'D TALK ABOUT FUNCTIONS AND THEIR RELATED CONCEPTS NEXT.
FUNCTIONS
IT'S ALL PRETTY ABSTRACT, BUT YOU'LL BE FINE AS LONG AS YOU TAKE YOUR TIME AND THINK HARD ABOUT EACH NEW IDEA.
GOT IT.
LET'S START BY DEFINING THE CONCEPT ITSELF.
SOUNDS GOOD.


IMAGINE THE FOLLOWING SCENARIO:
CAPTAIN ICHINOSE, IN A PLEASANT MOOD, DECIDES TO TREAT US FRESHMEN TO LUNCH.
SO WE FOLLOW HIM TO RESTAURANT A.
FOLLOW ME!
THIS IS THE RESTAURANT MENU.
UDON 500 YEN
CURRY 700 YEN
BREADED PORK 1000 YEN
BROILED EEL 1500 YEN
BUT THERE IS A CATCH, OF COURSE.
WHAT DO YOU MEAN?
SINCE HE'S THE ONE PAYING, HE GETS A SAY IN ANY AND ALL ORDERS.
?
KIND OF LIKE THIS:

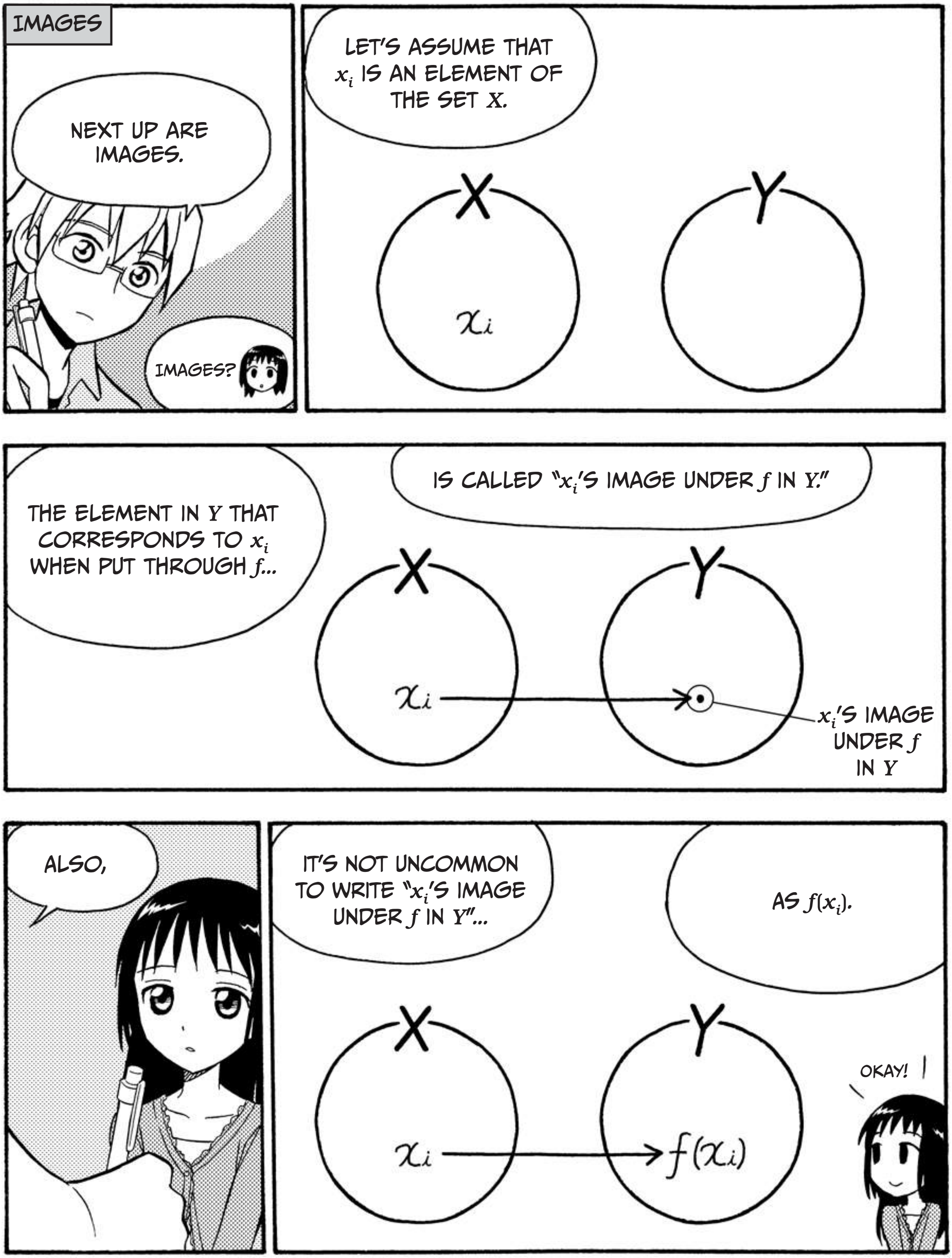

WE WOULDN'T REALLY BE ABLE TO SAY NO IF HE TOLD US TO ORDER THE CHEAPEST DISH, RIGHT?
UDON FOR EVERYONE!
YURINO
YOSHIDA
YAJIMA
TOMIYAMA
UDON
CURRY
BREADED PORK
BROILED EEL
OR SAY, IF HE JUST TOLD US ALL TO ORDER DIFFERENT THINGS.
ORDER DIFFERENT STUFF!
YURINO
YOSHIDA
YAJIMA
TOMIYAMA
UDON
CURRY
BREADED PORK
BROILED EEL

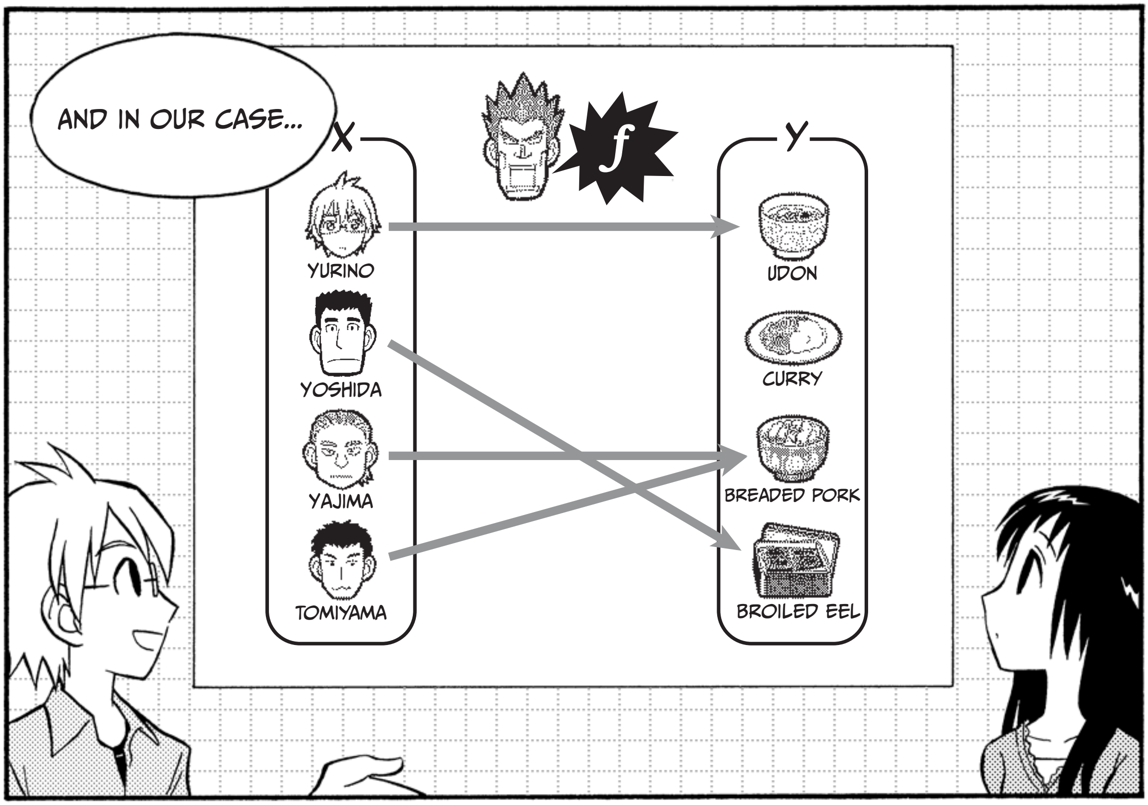

EVEN IF HE TOLD US TO ORDER OUR FAVORITES, WE WOULDN'T REALLY HAVE A CHOICE. THIS MIGHT MAKE US THE MOST HAPPY, BUT THAT DOESN'T CHANGE THE FACT THAT WE HAVE TO OBEY HIM.
ORDER WHAT YOU WANT!
YURINO
YOSHIDA
YAJIMA
TOMIYAMA
UDON
CURRY
BREADED PORK
BROILED EEL
YOU COULD SAY THAT THE CAPTAIN'S ORDERING GUIDELINES ARE LIKE A "RULE" THAT BINDS ELEMENTS OF X TO ELEMENTS OF Y.
RULE!
X
YURINO
?
YOSHIDA
?
YAJIMA
?
TOMIYAMA
?
Y
UDON
CURRY
BREADED PORK
BROILED EEL

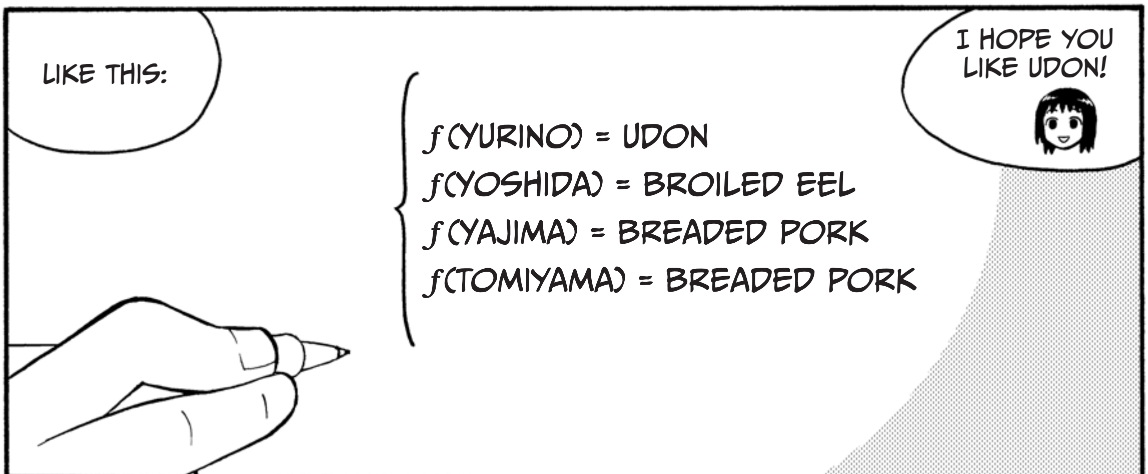


FUNCTIONS

A rule that binds elements of the set X to elements of the set Y is called "a *function* from X to Y." X is usually called the *domain* and Y the *co-domain* or *target set* of the function.

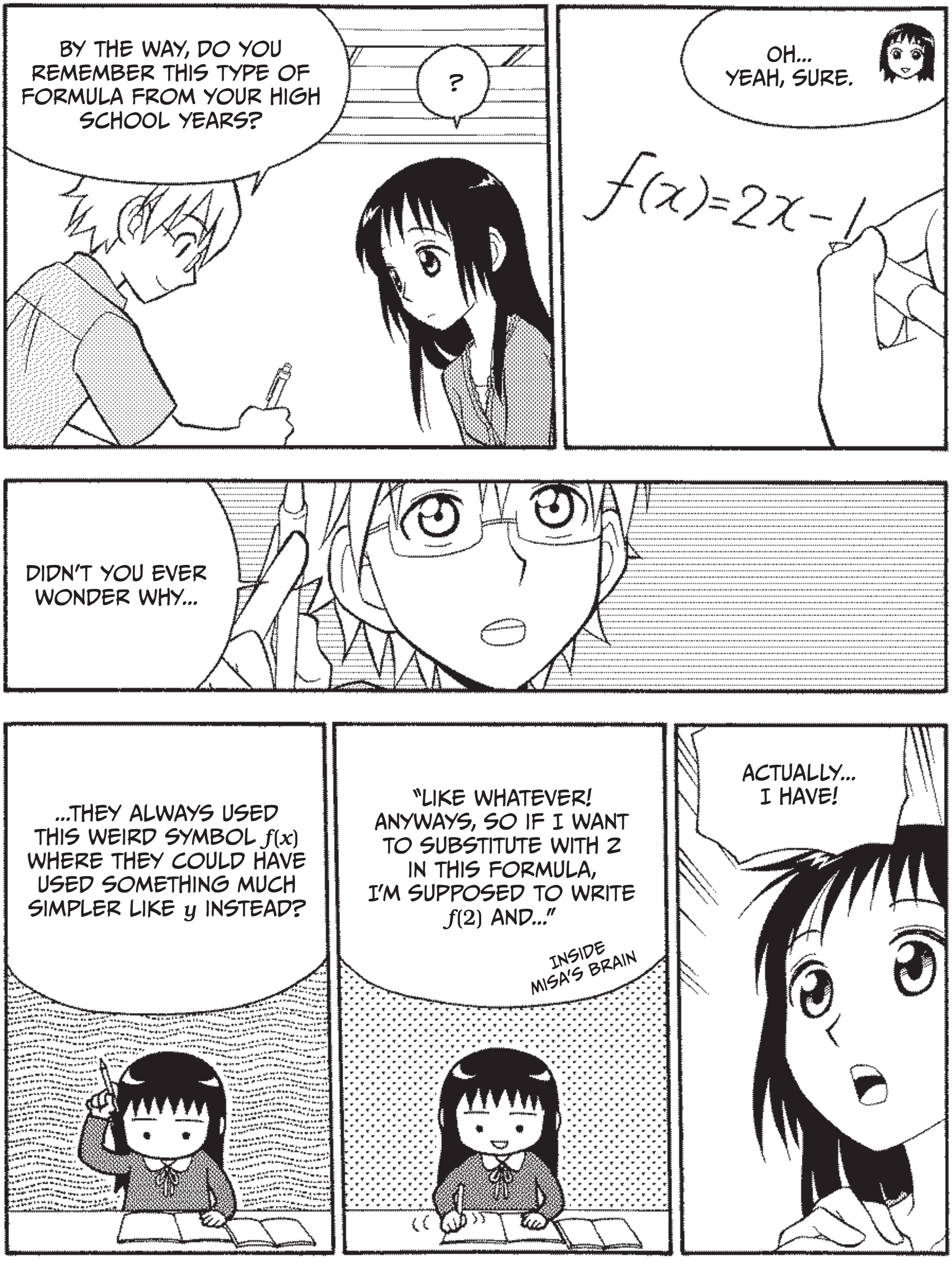

IMAGES
NEXT UP ARE IMAGES.
IMAGES?
LET'S ASSUME THAT x_i IS AN ELEMENT OF THE SET X.
X
x_i
Y
THE ELEMENT IN Y THAT CORRESPONDS TO x_i WHEN PUT THROUGH f...
IS CALLED "x_i'S IMAGE UNDER f IN Y."
X
x_i
Y
x_i'S IMAGE UNDER f IN Y
ALSO,
IT'S NOT UNCOMMON TO WRITE "x_i'S IMAGE UNDER f IN Y"...
AS $f(x_i)$.
X
x_i
Y
$f(x_i)$
OKAY!

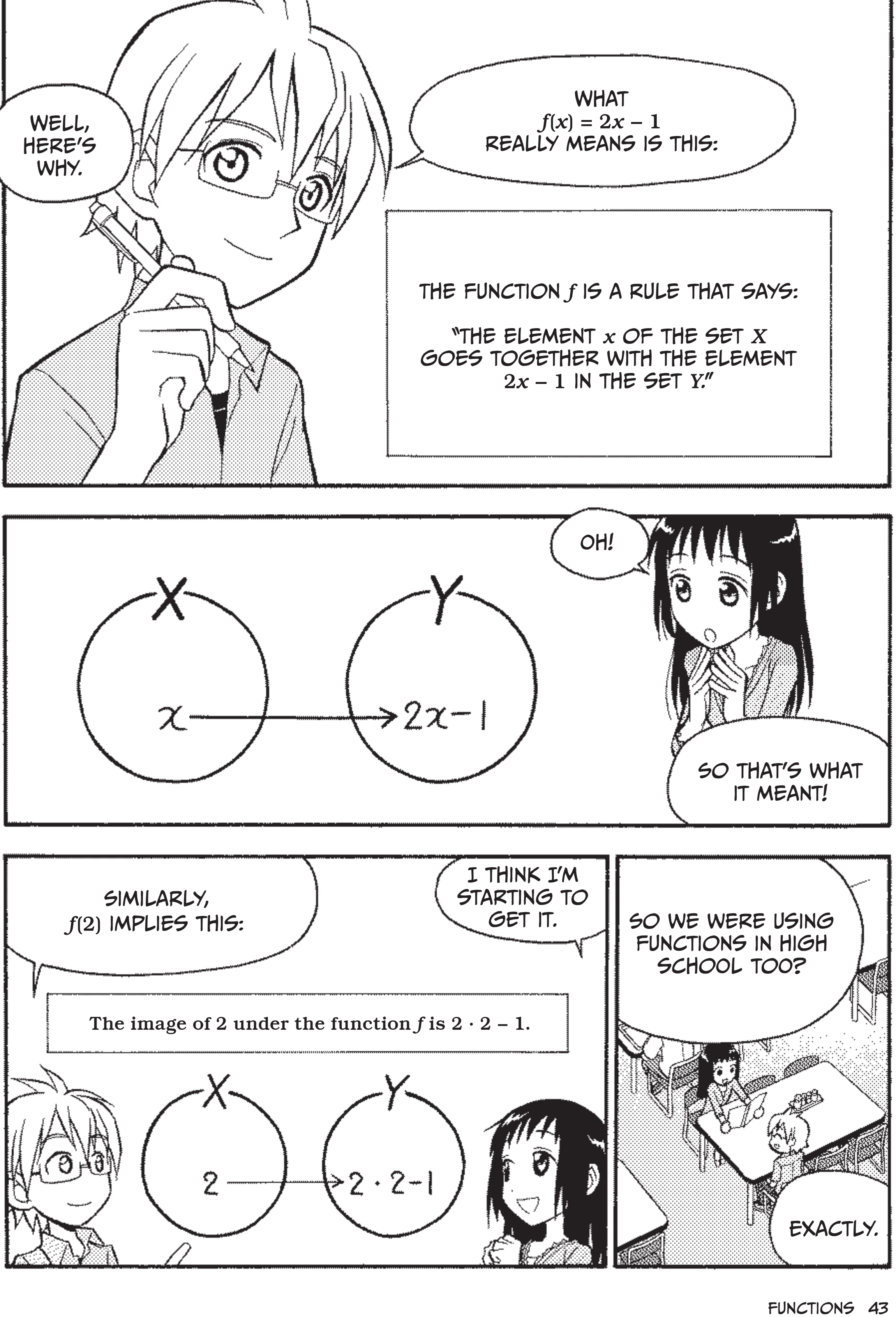


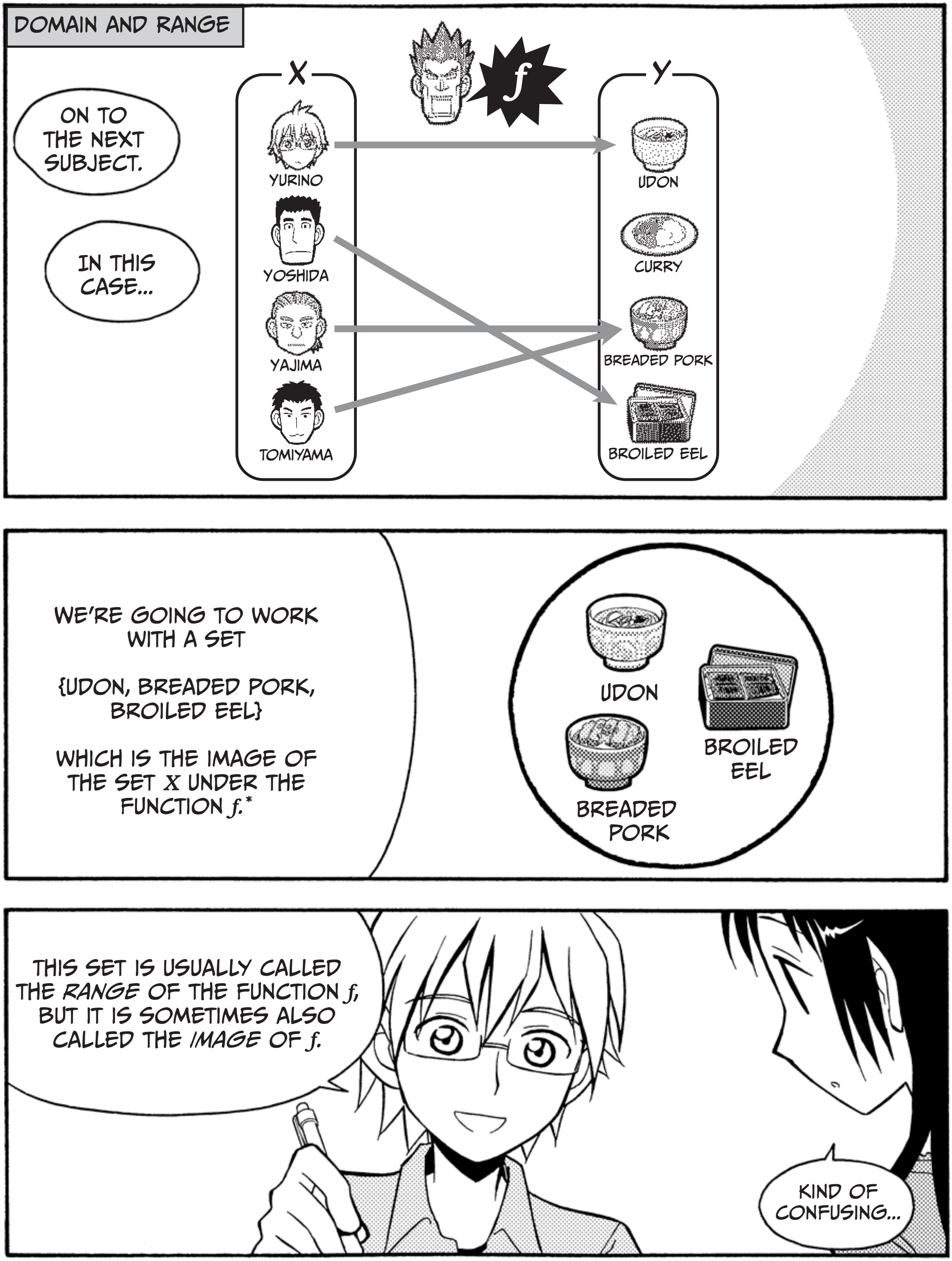


IMAGE

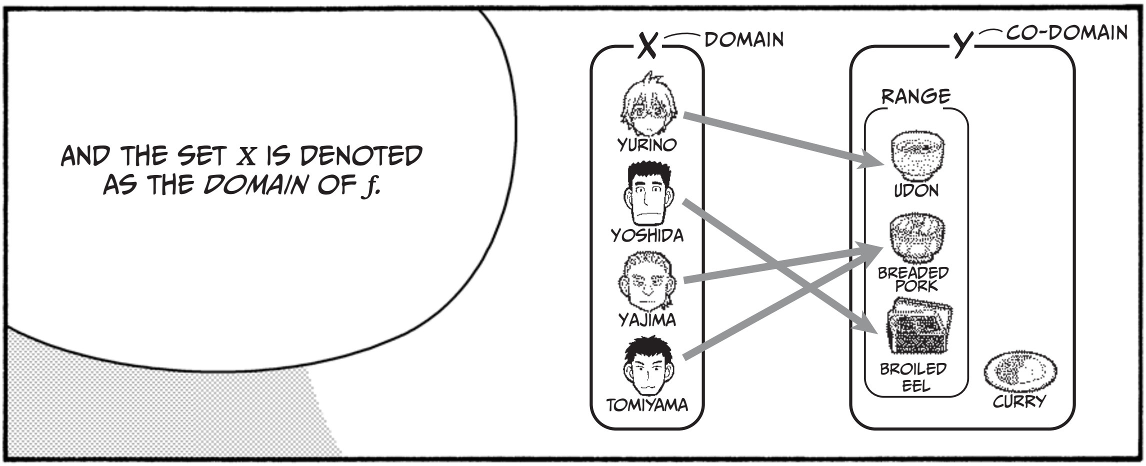
This is the element in Y that corresponds to x_i of the set X, when put through the function f.

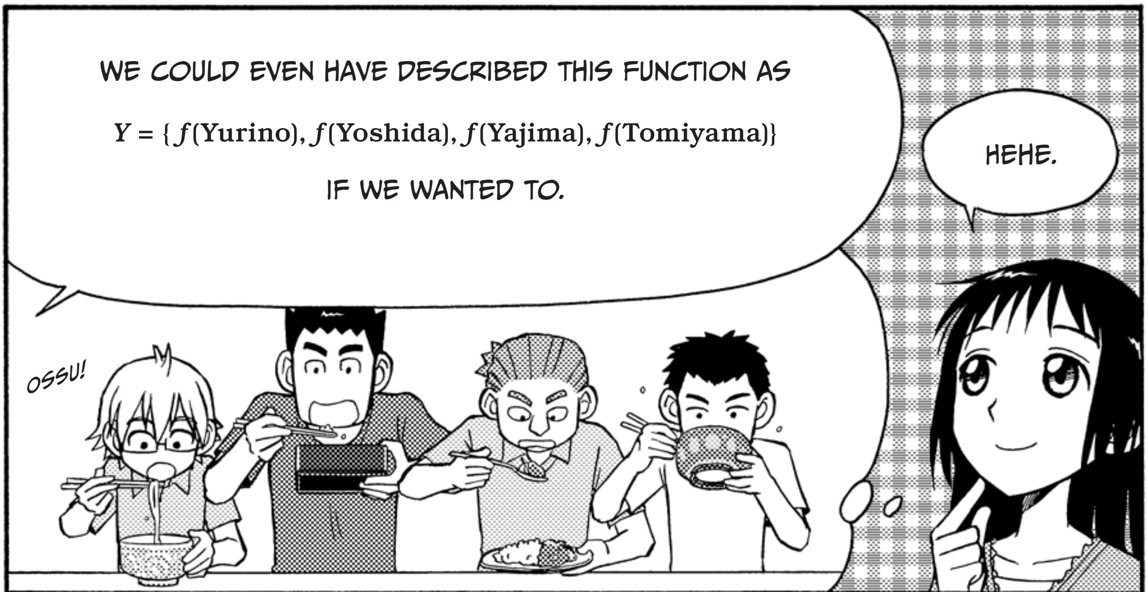

BY THE WAY, DO YOU REMEMBER THIS TYPE OF FORMULA FROM YOUR HIGH SCHOOL YEARS?
?
OH... YEAH, SURE.
f(x)=2x-1
DIDN'T YOU EVER WONDER WHY...
...THEY ALWAYS USED THIS WEIRD SYMBOL f(x) WHERE THEY COULD HAVE USED SOMETHING MUCH SIMPLER LIKE y INSTEAD?
"LIKE WHATEVER! ANYWAYS, SO IF I WANT TO SUBSTITUTE WITH 2 IN THIS FORMULA, I'M SUPPOSED TO WRITE f(2) AND..."
INSIDE MISA'S BRAIN
ACTUALLY... I HAVE!


WELL, HERE'S WHY.
WHAT $f(x) = 2x - 1$ REALLY MEANS IS THIS:
THE FUNCTION f IS A RULE THAT SAYS:
"THE ELEMENT x OF THE SET X GOES TOGETHER WITH THE ELEMENT $2x - 1$ IN THE SET Y."
OH!
X
Y
x → 2x−1
SO THAT'S WHAT IT MEANT!
SIMILARLY, $f(2)$ IMPLIES THIS:
I THINK I'M STARTING TO GET IT.
The image of 2 under the function f is $2 \cdot 2 - 1$.
X
Y
2 → 2 · 2−1
SO WE WERE USING FUNCTIONS IN HIGH SCHOOL TOO?
EXACTLY.



* THE TERM *IMAGE* IS USED HERE TO DESCRIBE THE *SET* OF ELEMENTS IN Y THAT ARE THE IMAGE OF AT LEAST ONE ELEMENT IN X.





Range and Co-Domain

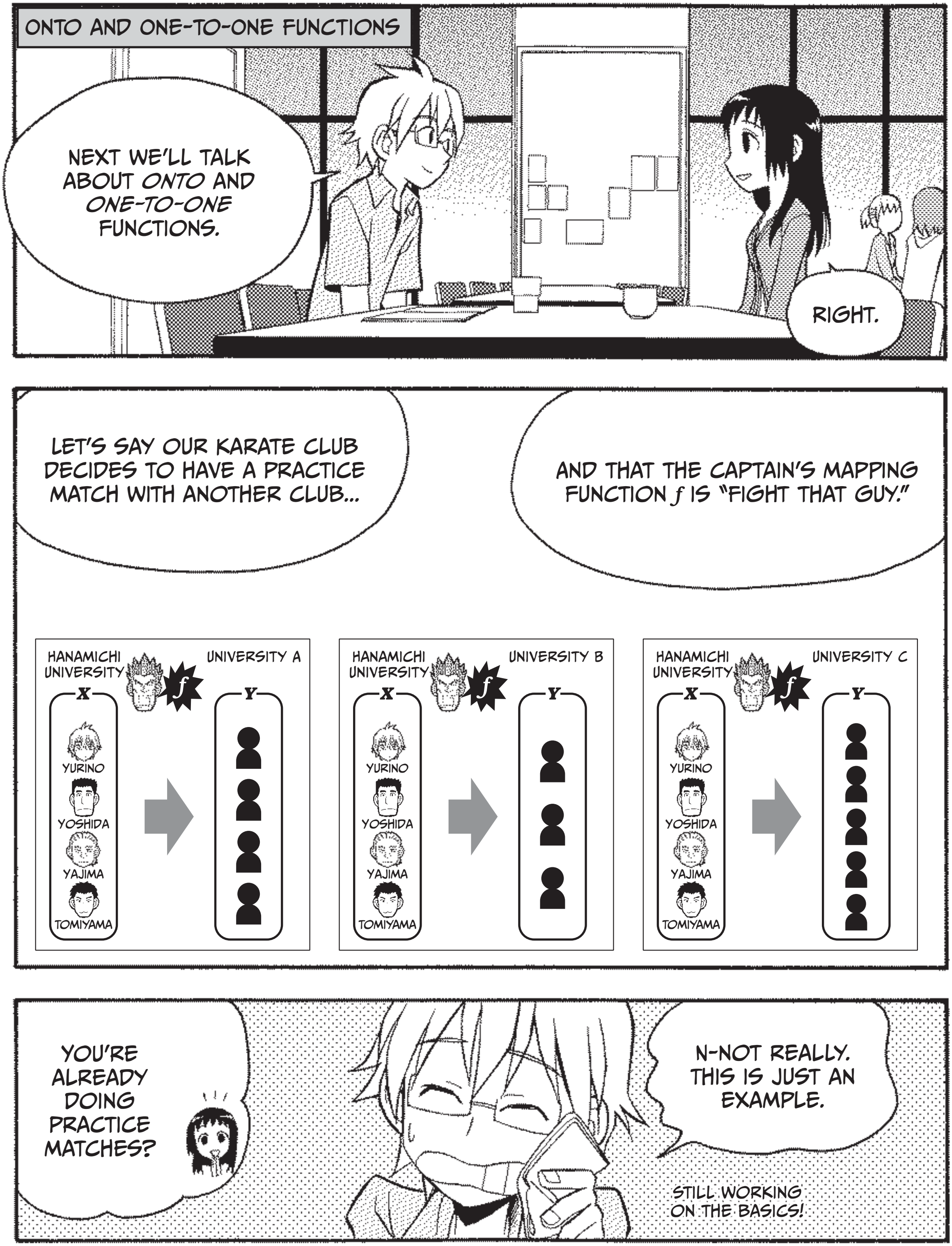
The set that encompasses the function f's image $\{f(x_1), f(x_2), \ldots, f(x_n)\}$ is called the *range* of f, and the (possibly larger) set being mapped into is called its *co-domain*.

The relationship between the range and the co-domain Y is as follows:

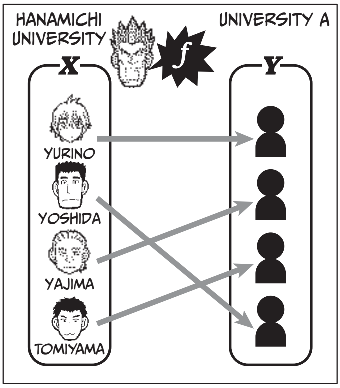
$$\{f(x_1), f(x_2), \ldots, f(x_n)\} \subset Y$$

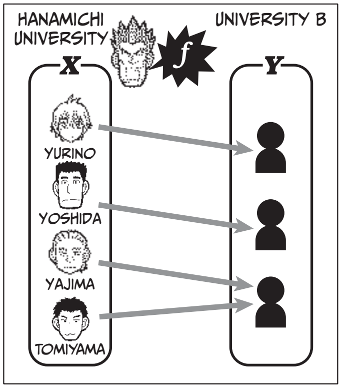
In other words, a function's range is a subset of its co-domain. In the special case where all elements in Y are an image of some element in X, we have

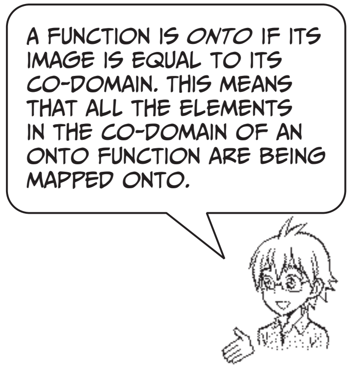
$$\{f(x_1), f(x_2), \ldots, f(x_n)\} = Y$$


ONTO AND ONE-TO-ONE FUNCTIONS
NEXT WE'LL TALK ABOUT *ONTO* AND *ONE-TO-ONE* FUNCTIONS.
RIGHT.
LET'S SAY OUR KARATE CLUB DECIDES TO HAVE A PRACTICE MATCH WITH ANOTHER CLUB...
AND THAT THE CAPTAIN'S MAPPING FUNCTION f IS "FIGHT THAT GUY."
HANAMICHI UNIVERSITY
UNIVERSITY A
f
X
Y
YURINO
YOSHIDA
YAJIMA
TOMIYAMA
HANAMICHI UNIVERSITY
UNIVERSITY B
f
X
Y
YURINO
YOSHIDA
YAJIMA
TOMIYAMA
HANAMICHI UNIVERSITY
UNIVERSITY C
f
X
Y
YURINO
YOSHIDA
YAJIMA
TOMIYAMA
YOU'RE ALREADY DOING PRACTICE MATCHES?
N-NOT REALLY. THIS IS JUST AN EXAMPLE.
STILL WORKING ON THE BASICS!

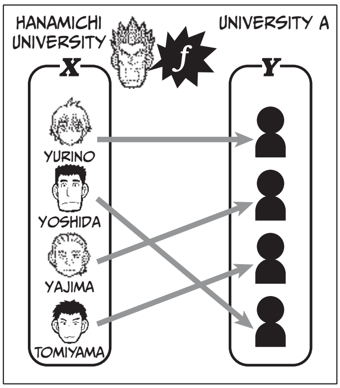
ONTO FUNCTIONS

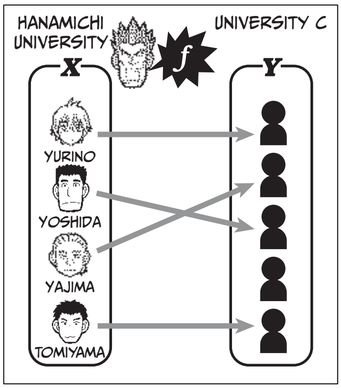


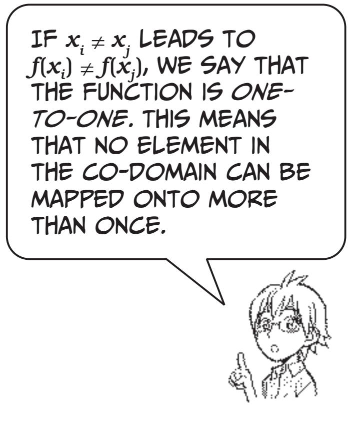




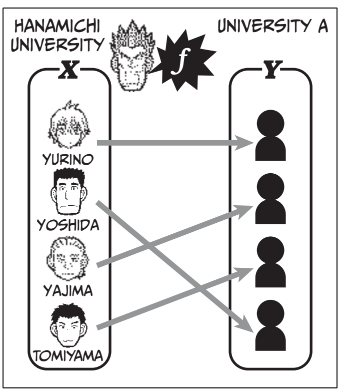
ONE-TO-ONE FUNCTIONS

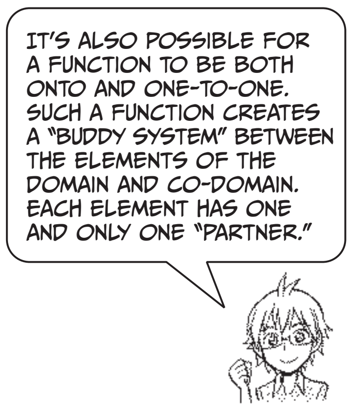


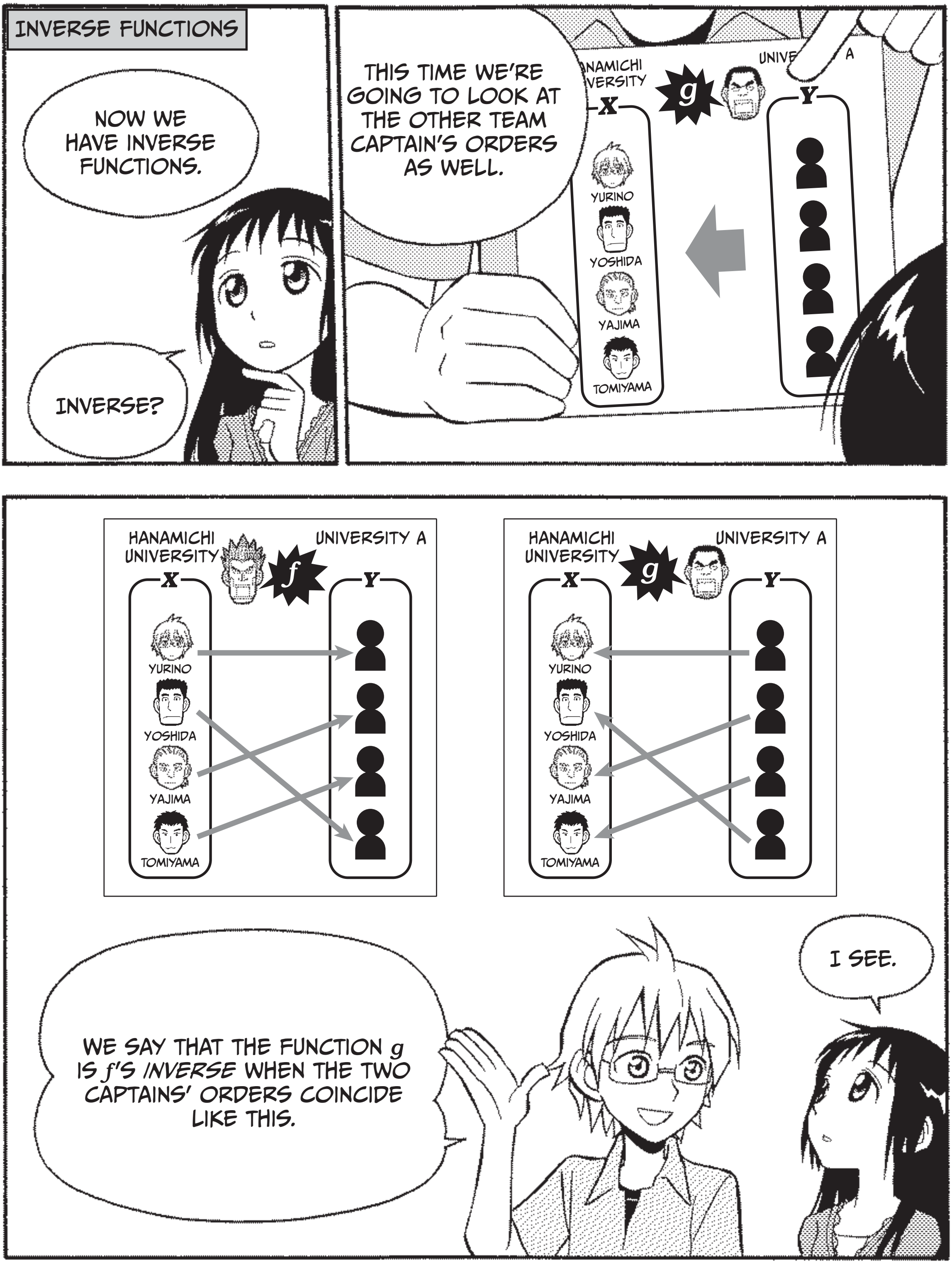


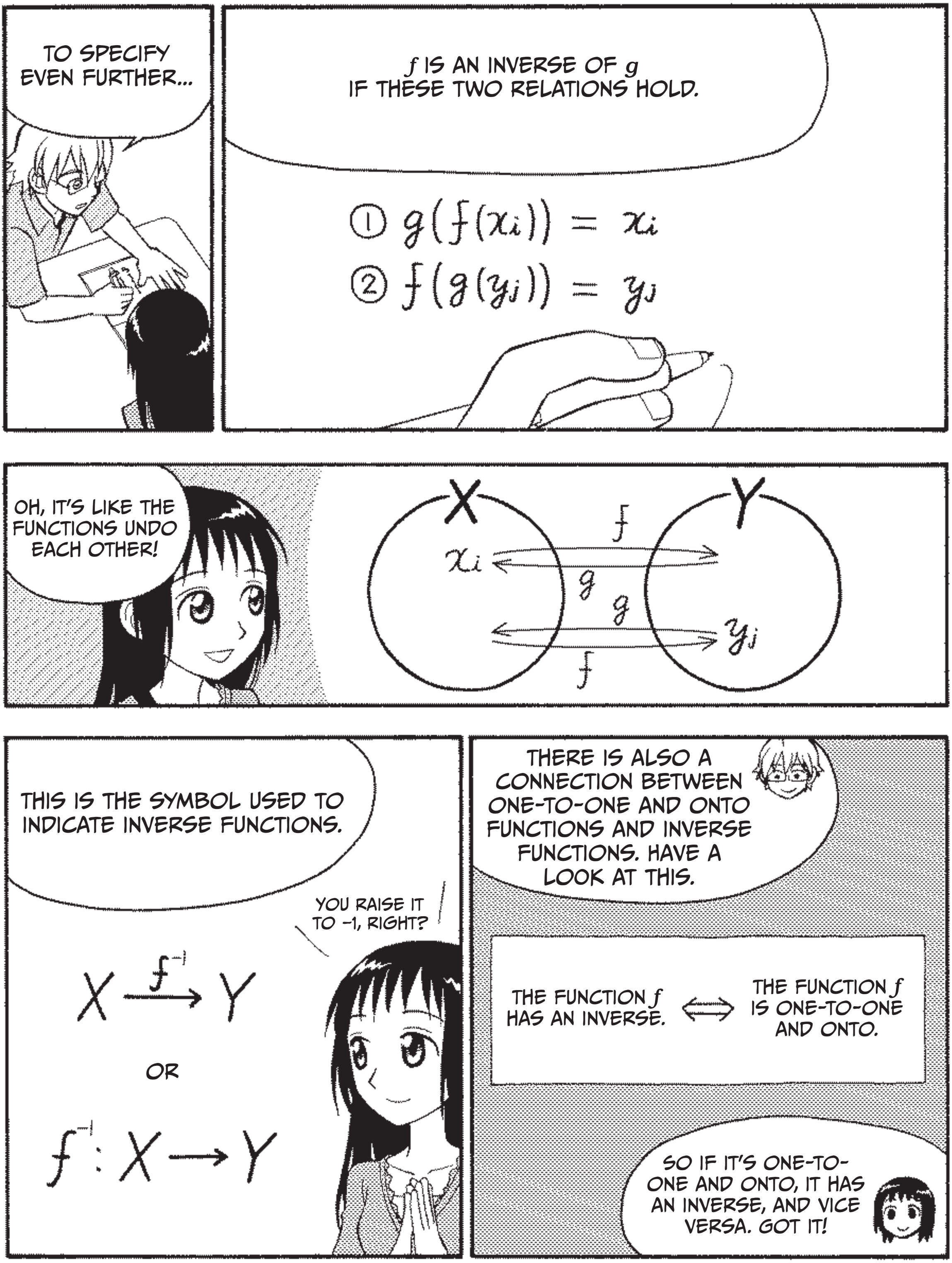


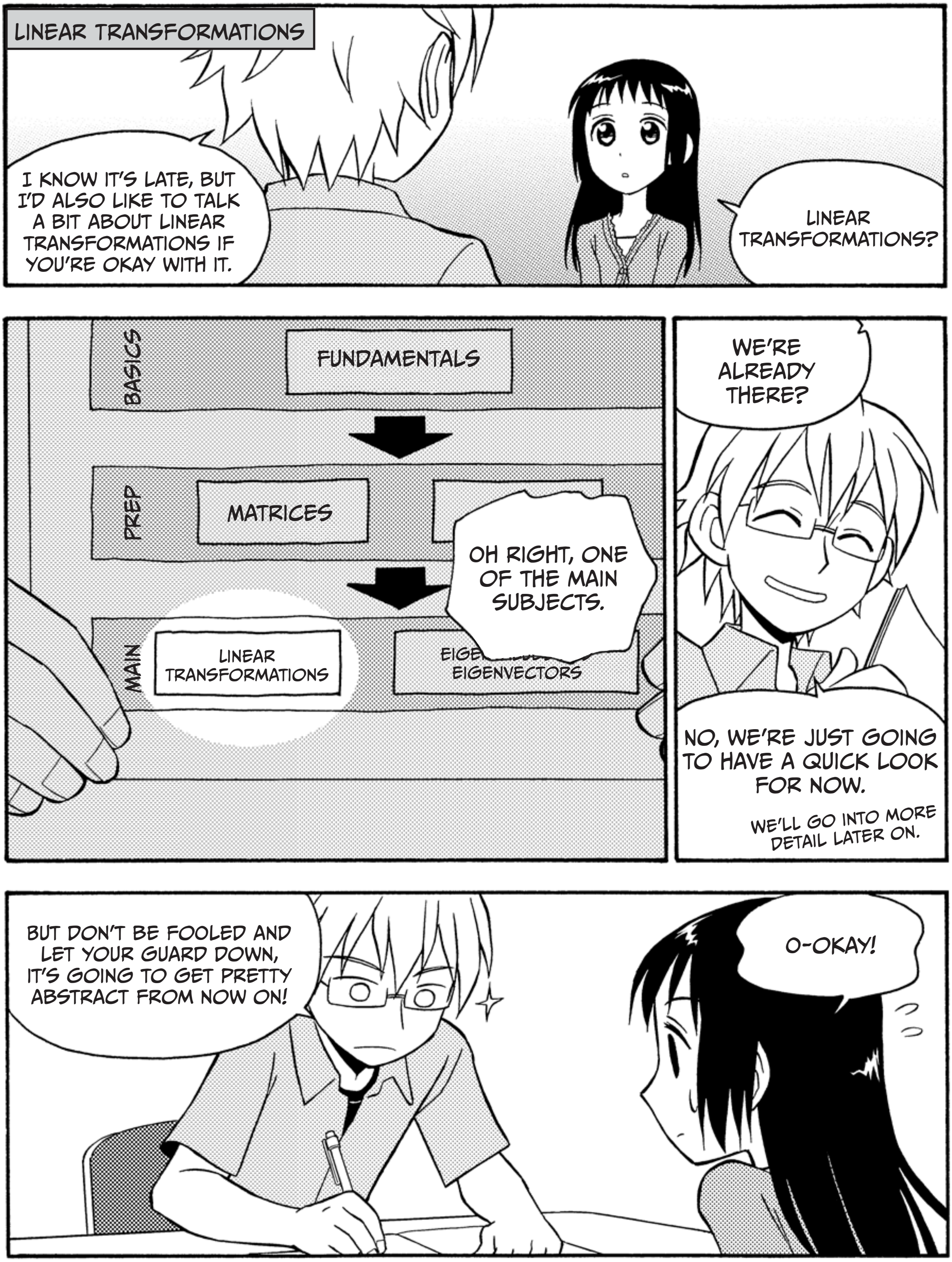
ONE-TO-ONE AND ONTO FUNCTIONS

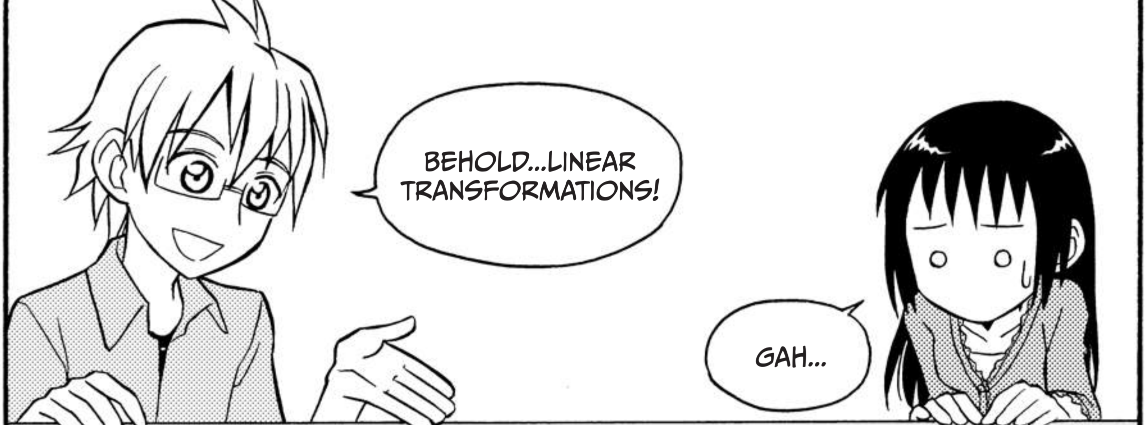





INVERSE FUNCTIONS
NOW WE HAVE INVERSE FUNCTIONS.
INVERSE?
THIS TIME WE'RE GOING TO LOOK AT THE OTHER TEAM CAPTAIN'S ORDERS AS WELL.
g
X
Y
YURINO
YOSHIDA
YAJIMA
TOMIYAMA
HANAMICHI UNIVERSITY
UNIVERSITY A
f
X
Y
YURINO
YOSHIDA
YAJIMA
TOMIYAMA
HANAMICHI UNIVERSITY
UNIVERSITY A
g
X
Y
YURINO
YOSHIDA
YAJIMA
TOMIYAMA
WE SAY THAT THE FUNCTION g IS f'S INVERSE WHEN THE TWO CAPTAINS' ORDERS COINCIDE LIKE THIS.
I SEE.


TO SPECIFY EVEN FURTHER...
f IS AN INVERSE OF g IF THESE TWO RELATIONS HOLD.
① $g(f(x_i)) = x_i$
② $f(g(y_j)) = y_j$
OH, IT'S LIKE THE FUNCTIONS UNDO EACH OTHER!
X
Y
f
x_i
g
g
y_j
f
THIS IS THE SYMBOL USED TO INDICATE INVERSE FUNCTIONS.
$X \xrightarrow{f^{-1}} Y$
OR
$f^{-1} : X \rightarrow Y$
YOU RAISE IT TO −1, RIGHT?
THERE IS ALSO A CONNECTION BETWEEN ONE-TO-ONE AND ONTO FUNCTIONS AND INVERSE FUNCTIONS. HAVE A LOOK AT THIS.
THE FUNCTION f HAS AN INVERSE. ⟺ THE FUNCTION f IS ONE-TO-ONE AND ONTO.
SO IF IT'S ONE-TO-ONE AND ONTO, IT HAS AN INVERSE, AND VICE VERSA. GOT IT!


LINEAR TRANSFORMATIONS
I KNOW IT'S LATE, BUT I'D ALSO LIKE TO TALK A BIT ABOUT LINEAR TRANSFORMATIONS IF YOU'RE OKAY WITH IT.
LINEAR TRANSFORMATIONS?
BASICS
FUNDAMENTALS
PREP
MATRICES
MAIN
LINEAR TRANSFORMATIONS
EIGENVECTORS
OH RIGHT, ONE OF THE MAIN SUBJECTS.
WE'RE ALREADY THERE?
NO, WE'RE JUST GOING TO HAVE A QUICK LOOK FOR NOW.
WE'LL GO INTO MORE DETAIL LATER ON.
BUT DON'T BE FOOLED AND LET YOUR GUARD DOWN, IT'S GOING TO GET PRETTY ABSTRACT FROM NOW ON!
O-OKAY!

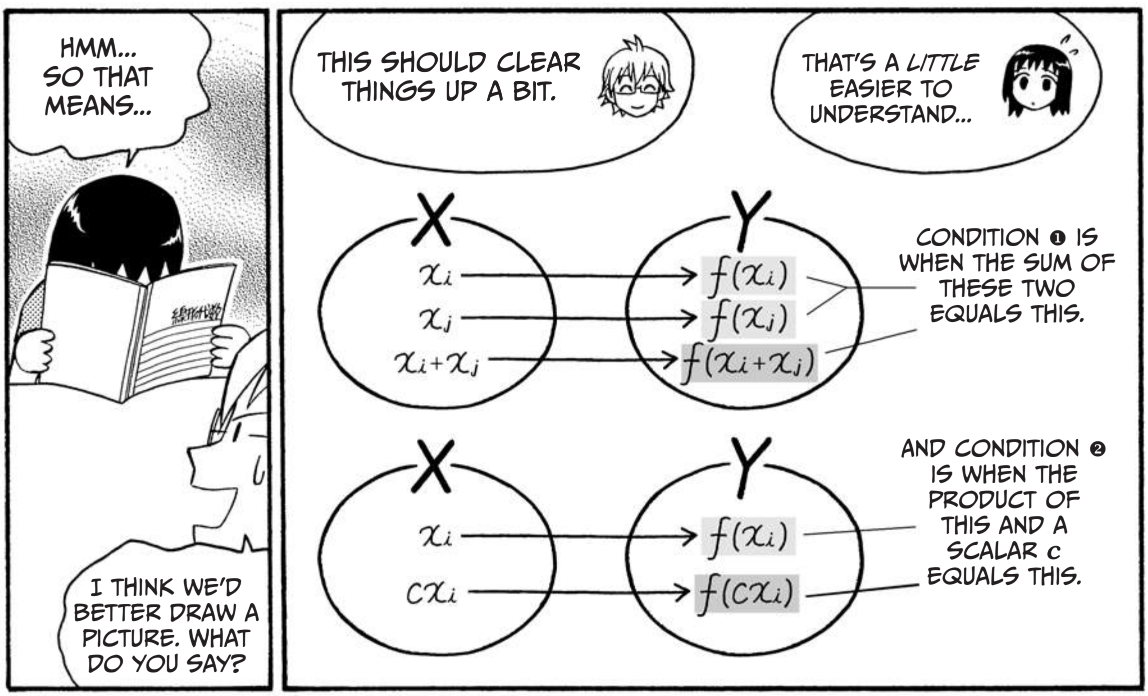


LINEAR TRANSFORMATIONS

Let x_i and x_j be two arbitrary elements of the set X, c be any real number, and f be a function from X to Y. f is called a *linear transformation* from X to Y if it satisfies the following two conditions:

❶ $f(x_i) + f(x_j) = f(x_i + x_j)$

❷ $cf(x_i) = f(cx_i)$



LET'S HAVE A LOOK AT A COUPLE OF EXAMPLES.

AN EXAMPLE OF A LINEAR TRANSFORMATION

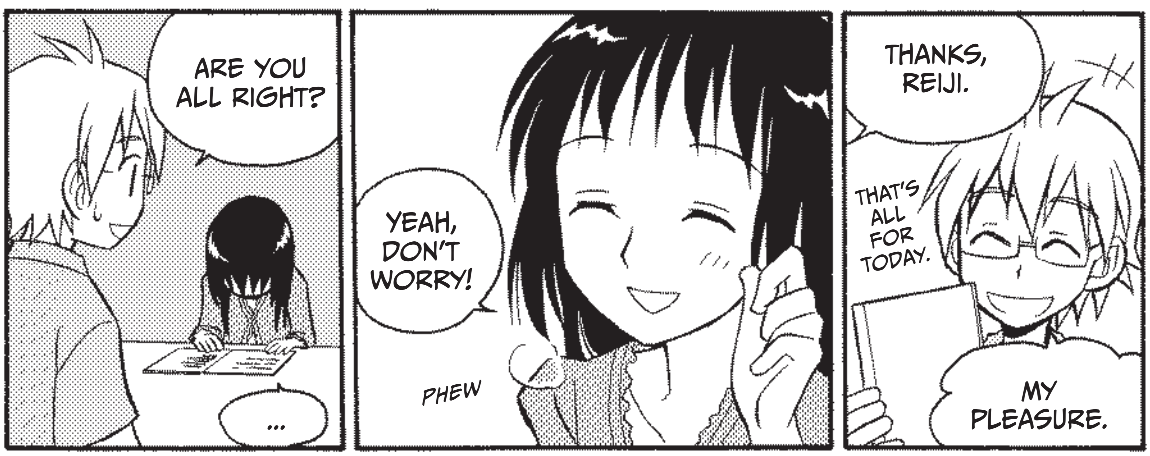
The function $f(x) = 2x$ is a linear transformation. This is because it satisfies both ❶ and ❷, as you can see in the table below.

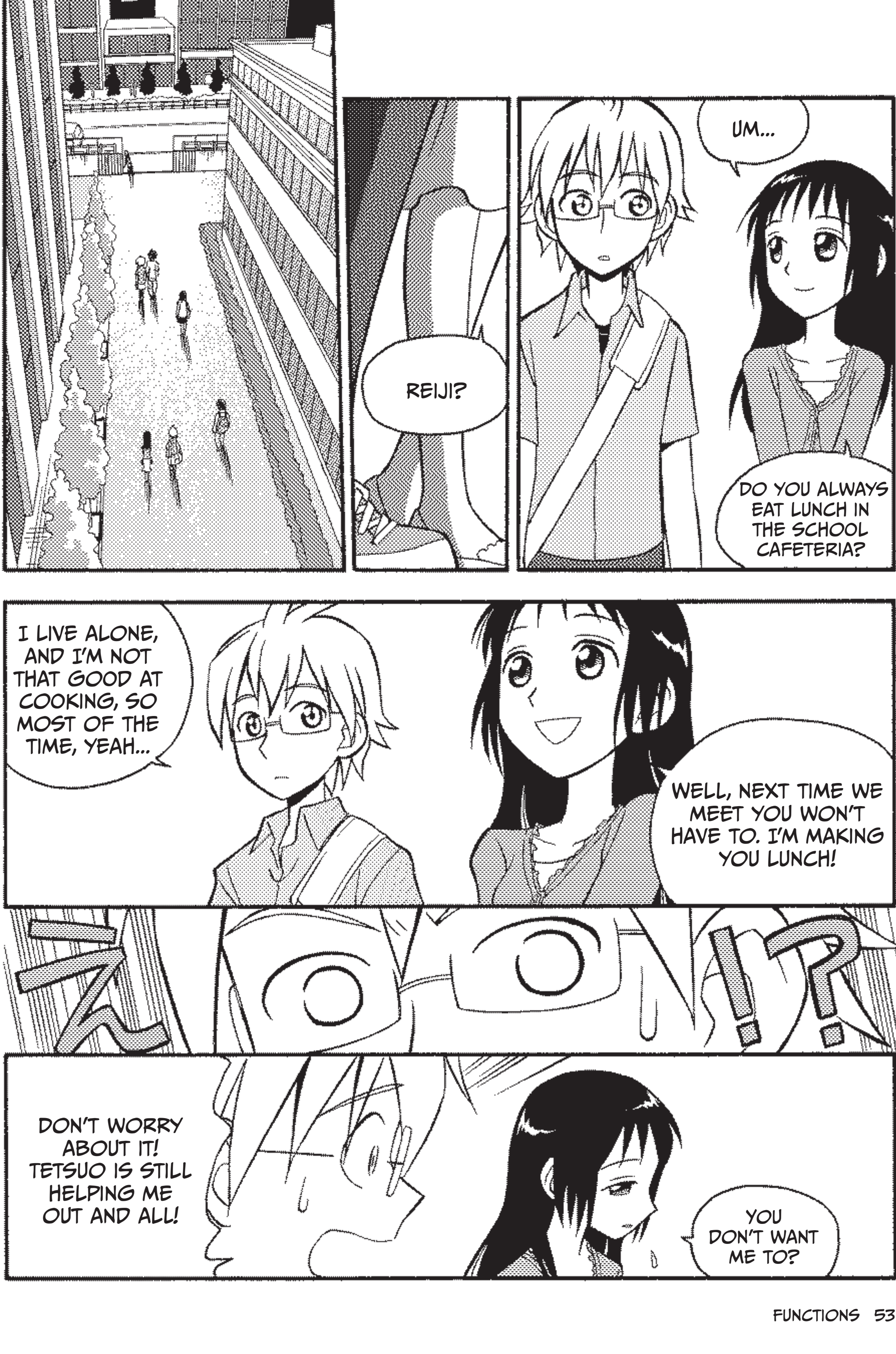
Condition ❶	$\begin{cases} f(x_i) + f(x_j) = 2x_i + 2x_j \\ f(x_i + x_j) = 2(x_i + x_j) = 2x_i + 2x_j \end{cases}$
Condition ❷	$\begin{cases} cf(x_i) = c(2x_i) = 2cx_i \\ f(cx_i) = 2(cx_i) = 2cx_i \end{cases}$

AN EXAMPLE OF A FUNCTION THAT IS NOT A LINEAR TRANSFORMATION

The function $f(x) = 2x - 1$ is not a linear transformation. This is because it satisfies neither ❶ nor ❷, as you can see in the table below.

Condition ❶	$\begin{cases} f(x_i) + f(x_j) = (2x_i - 1) + (2x_j - 1) = 2x_i + 2x_j - 2 \\ f(x_i + x_j) = 2(x_i + x_j) - 1 = 2x_i + 2x_j - 1 \end{cases}$
Condition ❷	$\begin{cases} cf(x_i) = c(2x_i - 1) = 2cx_i - c \\ f(cx_i) = 2(cx_i) - 1 = 2cx_i - 1 \end{cases}$




REIJI?
UM...
DO YOU ALWAYS EAT LUNCH IN THE SCHOOL CAFETERIA?
I LIVE ALONE, AND I'M NOT THAT GOOD AT COOKING, SO MOST OF THE TIME, YEAH...
WELL, NEXT TIME WE MEET YOU WON'T HAVE TO. I'M MAKING YOU LUNCH!
!?
YOU DON'T WANT ME TO?
DON'T WORRY ABOUT IT! TETSUO IS STILL HELPING ME OUT AND ALL!


NO, THAT'S NOT IT, IT'S JUST...
UH...
ON SECOND THOUGHT, I'D LOVE FOR YOU TO...
...MAKE ME LUNCH.
GREAT!
I MAKE A LOT OF THEM FOR MY BROTHER TOO, YOU KNOW—STAMINA LUNCHES.
OH...HOW LOVELY.

COMBINATIONS AND PERMUTATIONS

I thought the best way to explain combinations and permutations would be to give a concrete example.

I'll start by explaining the PROBLEM, then I'll establish a good WAY OF THINKING, and finally I'll present a SOLUTION.

PROBLEM

Reiji bought a CD with seven different songs on it a few days ago. Let's call the songs A, B, C, D, E, F, and G. The following day, while packing for a car trip he had planned with his friend Nemoto, it struck him that it might be nice to take the songs along to play during the drive. But he couldn't take all of the songs, since his taste in music wasn't very compatible with Nemoto's. After some deliberation, he decided to make a new CD with only three songs on it from the original seven.

Questions:

1. In how many ways can Reiji select three songs from the original seven?
2. In how many ways can the three songs be arranged?
3. In how many ways can a CD be made, where three songs are chosen from a pool of seven?

WAY OF THINKING

It is possible to solve question 3 by dividing it into these two subproblems:

1. Choose three songs out of the seven possible ones.
2. Choose an order in which to play them.

As you may have realized, these are the first two questions. The solution to question 3, then, is as follows:

SOLUTION TO QUESTION 1 ·	SOLUTION TO QUESTION 2 =	SOLUTION TO QUESTION 3
In how many ways can Reiji select three songs from the original seven?	In how many ways can the three songs be arranged?	In how many ways can a CD be made, where three songs are chosen from a pool of seven?

SOLUTION

1. **In how many ways can Reiji select three songs from the original seven?**

 All 35 different ways to select the songs are in the table below. Feel free to look them over.

Pattern 1	A and B and C
Pattern 2	A and B and D
Pattern 3	A and B and E
Pattern 4	A and B and F
Pattern 5	A and B and G
Pattern 6	A and C and D
Pattern 7	A and C and E
Pattern 8	A and C and F
Pattern 9	A and C and G
Pattern 10	A and D and E
Pattern 11	A and D and F
Pattern 12	A and D and G
Pattern 13	A and E and F
Pattern 14	A and E and G
Pattern 15	A and F and G
Pattern 16	B and C and D
Pattern 17	B and C and E
Pattern 18	B and C and F
Pattern 19	B and C and G
Pattern 20	B and D and E
Pattern 21	B and D and F
Pattern 22	B and D and G
Pattern 23	B and E and F
Pattern 24	B and E and G
Pattern 25	B and F and G
Pattern 26	C and D and E
Pattern 27	C and D and F
Pattern 28	C and D and G
Pattern 29	C and E and F
Pattern 30	C and E and G
Pattern 31	C and F and G
Pattern 32	D and E and G
Pattern 33	D and E and G
Pattern 34	D and F and G
Pattern 35	E and F and G

Choosing k among n items without considering the order in which they are chosen is called a *combination*. The number of different ways this can be done is written by using the binomial coefficient notation:

$$\binom{n}{k}$$

which is read "n choose k."

In our case,

$$\binom{7}{3} = 35$$

2. In how many ways can the three songs be arranged?

Let's assume we chose the songs A, B, and C. This table illustrates the 6 different ways in which they can be arranged:

Song 1	Song 2	Song 3
A	B	C
A	C	B
B	A	C
B	C	A
C	A	B
C	B	A

Suppose we choose B, E, and G instead:

Song 1	Song 2	Song 3
B	E	G
B	G	E
E	B	G
E	G	B
G	B	E
G	E	B

Trying a few other selections will reveal a pattern: The number of possible arrangements does not depend on which three elements we choose—there are always six of them. Here's why:

Our result (6) can be rewritten as $3 \cdot 2 \cdot 1$, which we get like this:

1. We start out with all three songs and can choose any one of them as our first song.
2. When we're picking our second song, only two remain to choose from.
3. For our last song, we're left with only one choice.

   This gives us 3 possibilities · 2 possibilities · 1 possibility = 6 possibilities.

3. In how many ways can a CD be made, where three songs are chosen from a pool of seven?

The different possible patterns are

The number of ways to choose three songs from seven	·	The number of ways the three songs can be arranged

$$= \binom{7}{3} \cdot 6$$

$$= 35 \cdot 6$$

$$= 210$$

This means that there are 210 different ways to make the CD.

Choosing three from seven items in a certain order creates a *permutation* of the chosen items. The number of possible permutations of k objects chosen among n objects is written as

$${}_nP_k$$

In our case, this comes to

$${}_7P_3 = 210$$

The number of ways n objects can be chosen among n possible ones is equal to n-factorial:

$${}_nP_n = n! = n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 2 \cdot 1$$

For instance, we could use this if we wanted to know how many different ways seven objects can be arranged. The answer is

$$7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040$$

I've listed all possible ways to choose three songs from the seven original ones (A, B, C, D, E, F, and G) in the table below.

	Song 1	Song 2	Song 3
Pattern 1	A	B	C
Pattern 2	A	B	D
Pattern 3	A	B	E
...	...	...	...
Pattern 30	A	G	F
Pattern 31	B	A	C
...	...	...	...
Pattern 60	B	G	F
Pattern 61	C	A	B
...	...	...	...
Pattern 90	C	G	F
Pattern 91	D	A	B
...	...	...	...
Pattern 120	D	G	F
Pattern 121	E	A	B
...	...	...	...
Pattern 150	E	G	F
Pattern 151	F	A	B
...	...	...	...
Pattern 180	F	G	E
Pattern 181	G	A	B
...	...	...	...
Pattern 209	G	E	F
Pattern 210	G	F	E

We can, analogous to the previous example, rewrite our problem of counting the different ways in which to make a CD as $7 \cdot 6 \cdot 5 = 210$. Here's how we get those numbers:

1. We can choose any of the **7** songs A, B, C, D, E, F, and G as our first song.
2. We can then choose any of the **6** remaining songs as our second song.
3. And finally we choose any of the now **5** remaining songs as our last song.

The definition of the binomial coefficient is as follows:

$$\binom{n}{r} = \frac{n\cdot(n-1)\cdots(n-(r-1))}{r\cdot(r-1)\cdots 1} = \frac{n\cdot(n-1)\cdots(n-r+1)}{r\cdot(r-1)\cdots 1}$$

Notice that

$$\begin{aligned}\binom{n}{r} &= \frac{n\cdot(n-1)\cdots(n-(r-1))}{r\cdot(r-1)\cdots 1} \\ &= \frac{n\cdot(n-1)\cdots(n-(r-1))}{r\cdot(r-1)\cdots 1}\cdot\frac{(n-r)\cdot(n-r+1)\cdots 1}{(n-r)\cdot(n-r+1)\cdots 1} \\ &= \frac{n\cdot(n-1)\cdots(n-(r-1))\cdot(n-r)\cdot(n-r+1)\cdots 1}{(r\cdot(r-1)\cdots 1)\cdot((n-r)\cdot(n-r+1)\cdots 1)} \\ &= \frac{n!}{r!\cdot(n-r)!}\end{aligned}$$

Many people find it easier to remember the second version:

$$\binom{n}{r} = \frac{n!}{r!\cdot(n-r)!}$$

We can rewrite question 3 (how many ways can the CD be made?) like this:

$$_7P_3 = \binom{7}{3}\cdot 6 = \binom{7}{3}\cdot 3! = \frac{7!}{3!\cdot 4!}\cdot 3! = \frac{7!}{4!} = \frac{7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1}{4\cdot3\cdot2\cdot1} = 7\cdot6\cdot5 = 210$$

NOT ALL "RULES FOR ORDERING" ARE FUNCTIONS

We talked about the three commands "Order the cheapest one!" "Order different stuff!" and "Order what you want!" as functions on pages 37–38. It is important to note, however, that "Order different stuff!" isn't actually a function in the strictest sense, because there are several different ways to obey that command.